Math 128

Final Examination – December 12, 2008

Name ______________________

8 problems, 100 points.

Instructions: Show all work – partial credit will be given, and "Answers without work are worth credit without points." You don't have to simplify your answers. You may use a simple calculator that is not graphing or programmable. You may have up to four 3x5 cards, but no other notes.

1. (a) *(6 points) Calculate f_{xy}, where $f(x, y) = 3e^{-2xy}$.*

   $f_x = 3 \cdot (-2y) \cdot e^{-2xy} = -6ye^{-2xy}$, so

$$f_{xy} = -6y \cdot (-2x) \cdot e^{-2xy} - 6e^{-2xy} = 12xye^{-2xy} - 6e^{-2xy}.$$

   (b) *(7 points) Evaluate the double integral $\int_0^1 \int_{x^2}^{\sqrt{x}} xy\, dydx$.*

$$\begin{aligned} \int_0^1 \left[x \cdot \frac{y^2}{2} \right]_{x^2}^{\sqrt{x}} dx &= \frac{1}{2} \int_0^1 x^2 - x^5\, dx \\ &= \frac{1}{2} \left[\frac{x^3}{3} - \frac{x^6}{6} \right]_0^1 \\ &= \frac{1}{2} (\frac{1}{3} - \frac{1}{6}) = \frac{1}{12}. \end{aligned}$$

   (c) *(7 points) Solve the differential equation $y' = -xy$.*

   We separate variables:

$$\frac{dy}{dx} = -xy \implies \frac{1}{y}\, dy = -x\, dx.$$

   We then integrate:

$$\begin{aligned} \int \frac{1}{y}\, dy &= \int -x\, dx \\ \ln |y| &= -\frac{1}{2}x^2 + C \\ |y| &= e^{-\frac{1}{2}x^2 + C} \\ y &= \pm e^C e^{-\frac{1}{2}x^2}. \end{aligned}$$

If we wanted, we could rewrite $A = \pm e^C$, so that $y = Ae^{-\frac{1}{2}x^2}$.

2. *Let X be a normally distributed random variable, with expected value* 3 *and variance 4.*

   (a) *(3 points) Set up an integral representing* $\Pr(2 \le X \le 5)$.

$$\int_2^5 \frac{1}{2\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x-3}{2})^2}\, dx.$$

   (b) *(4 points) Show how to use integration by substitution to evaluate your integral from part (a). Use of the table of areas under the standard normal distribution may be helpful.*
   We take $u = \dfrac{x-3}{2}$, so that $du = \dfrac{1}{2}dx$. Then

$$\begin{aligned}\int_2^5 \frac{1}{2\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x-3}{2})^2}\, dx &= \int_{u(2)}^{u(5)} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2}\, du \\ &= \int_{-0.5}^{1} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2}\, du \\ &= A(1) - A(-0.5) = A(1) + A(0.5).\end{aligned}$$

   You can look up the values of $A(1)$ and $A(0.5)$ on the included table.

3. (a) *(8 points) Let $f(x) = e^x \sin x$. Find an upper bound A for* $|f''(x)| = \left|\dfrac{d^2}{dx^2} e^x \sin x\right|$ *on the interval* $[-1, 2]$.
   We first calculate f'':

$$\begin{aligned}f'(x) &= e^x \sin x + e^x \cos x \\ f''(x) &= (e^x \sin x + e^x \cos x) + (e^x \cos x - e^x \sin x) \\ &= 2e^x \cos x.\end{aligned}$$

   Then $|f''(x)| = 2|e^x|\,|\cos x|$, and since $0 \le e^x \le e^2$ on the given interval, and $|\cos x| \le 1$ for all x, we find that $A = 2e^2$ is an upper bound.

   (b) *(4 points) Using your result from part (a), find an n so that the midpoint rule approximation M_n is accurate to* 0.01 *for the integral*

$$\int_{-1}^{2} e^x \sin x \, dx.$$

We use the error bound formula:

$$\text{error} M_n \le \frac{2e^2 \cdot (2-(-1))^3}{24n^2} = \frac{e^2 \cdot 27}{24n^2} = \frac{e^2 \cdot 9}{8n^2}.$$

So if

$$\frac{e^2 \cdot 9}{8n^2} \le 0.01 = \frac{1}{100},$$

then we have the desired accuracy. This means that

$$\frac{e^2 \cdot 9 \cdot 100}{8} \le n^2, \quad \text{or} \quad \frac{30e}{2\sqrt{2}} = \frac{15e}{\sqrt{2}} \le n.$$

Any n greater than $15e/\sqrt{2}$ has small enough error, for example $n = 30$.

4. *The random variable X has outcomes between 0 and π. The probability density function of X is $k \sin x$ for some constant k.*

   (a) *(7 points) Find k.*
   Since $k \sin x$ is a probability density function, and the outcomes of X are on $[0, \pi]$, we have that

   $$\int_0^{\pi} k \sin x \, dx = k \left[-\cos x\right]_0^{\pi} = 2k = 1.$$

   Thus $k = \frac{1}{2}$.

   (b) *(3 points) Set up an integral for $E(X)$.*
   (If you have trouble solving part (a), it's ok to leave your answer in terms of k.)

   $$E(X) = \int_0^{\pi} x \cdot \frac{1}{2} \sin x \, dx.$$

   (c) *(6 points) Evaluate your integral from part (b) to calculate $E(X)$.*
   We use integration by parts. The rule of thumb LATE tells us to

take $u = x$, so that $dv = \frac{1}{2}\sin x\,dx$, and $du = dx$, $v = -\frac{1}{2}\cos x$. We get

$$\begin{aligned}
&= \left[x\cdot(-\frac{1}{2}\cos x)\right]_0^{\pi} - \int_0^{\pi} -\frac{1}{2}\cos x\,dx \\
&= \left(\pi\cdot(-\frac{1}{2})\cdot(-1) - 0\cdot(-\frac{1}{2})\cdot 1\right) + \frac{1}{2}\left[\sin x\right]_0^{\pi} \\
&= \frac{\pi}{2}
\end{aligned}$$

5. *(16 points) Let* $f(x,y) = x^2 + 4y^3 - 6xy + 10$.

(a) *Find the partial derivatives* $\frac{\partial f}{\partial x}$ *and* $\frac{\partial f}{\partial y}$.

$\frac{\partial f}{\partial x} = 2x - 6y$, $\frac{\partial f}{\partial y} = 12y^2 - 6x$.

(b) *Find the critical points for* f.

We set $2x - 6y = 0$ and $12y^2 - 6x = 0$. Dividing the first equation by 2 and moving y to the rights gives us $x = 3y$. Substituting this into the other equation gives us $12y - 6x = 12y^2 - 18y = 0$. We factor this last equation as $6y(2y-3) = 0$. Thus, $y = 0$ or $y = \frac{3}{2}$, and since $x = 3y$, we have the critical points $(0,0)$ and $(\frac{9}{2}, \frac{3}{2})$.

(c) *Calculate the 2nd derivatives* $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$, *and* $\frac{\partial^2 f}{\partial x \partial y}$.

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}2x - 6y = 2$$
$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}12y^2 - 6x = -6$$
$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}12y^2 - 6x = 24y.$$

(d) *Using the 2nd derivative test, determine which points are relative maxima, relative minima, and saddle points.*

From the calculation of the 2nd derivative, we have the discriminant

$$D(x,y) = f_{xx}f_{yy} - (f_{xy})^2 = 48y - 36.$$

Plugging in the critical points, we get that
$D(0,0) = 0 - 36 = -36 < 0$, hence $(0,0)$ is a saddle point.

$D(\frac{9}{2}, \frac{3}{2}) = 48 \cdot \frac{3}{2} - 36 = 36 > 0$, and since $f_{xx} = 2 > 0$, that $(\frac{9}{2}, \frac{3}{2})$ is a relative minimum.

6.

(a) *(8 points) Show how to find a Taylor series for* $F(x) = \displaystyle\int_0^x \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2}\, dt.$

Hint: $F(x)$ *is an anti-derivative of* $\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$.

We first find a Taylor series for $\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$. Since

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!},$$

we get that

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} = \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{(-\frac{1}{2}x^2)^k}{k!} = \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{(-\frac{1}{2})^k x^{2k}}{k!}.$$

We integrate to get

$$F(x) = \int \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{(-\frac{1}{2})^k x^{2k}}{k!}\, dx = \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \int \frac{(-\frac{1}{2})^k x^{2k}}{k!}\, dx$$

$$= C + \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{(-\frac{1}{2})^k x^{2k+1}}{(2k+1)k!}.$$

Since $F(0) = 0$, we find that $0 = C + \sum 0 = C$, and that

$$F(x) = \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{(-\frac{1}{2})^k x^{2k+1}}{(2k+1)k!}.$$

(b) *(3 points) Let X be a random variable with the standard normal distribution. Using part (a), write down an infinite series for the probability* $\Pr(0 \le X \le 2)$.

Plug into the result from part (a):

$$\int_0^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\, dx = F(2) = \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{(-\frac{1}{2})^k 2^{2k+1}}{(2k+1)k!}.$$

7. *A certain lecturer often fails to turn off his mobile phone when teaching. Let the random variables X be the amount of lecture time (in hours) before his phone next rings.*

   (a) *(2 points) In 2-3 sentences, explain why X is exponentially distributed.*
   A number of answers received full credit.
   Because X is the amount of time between rare random events, and the events are memoryless.
   Because X is the amount of time between phone calls, like the examples we did in class.
   etc.

   (b) *(4 points) From experimental data, the expected value of X is 30 hours of classtime. The standard deviation is also 30 hours. What is the probability density function for X?*
   From class, an exponential random variable with probability density function ke^{-kx} has expected value $\frac{1}{k}$. Thus, $k = \frac{1}{30}$ in this example, and the probability density function is

$$\frac{1}{30}e^{-\frac{1}{30}x}.$$

   (c) *(4 points) The random variable Y is normally distributed, but also has expected value and standard deviation of 30. What is the probability density function for Y?*

$$\frac{1}{30\sqrt{2\pi}}e^{-\frac{1}{2}(\frac{x-30}{30})^2}.$$

8. *(8 points) A 100 L tank of water has two hoses flowing into it. One carries 3 L/min of saltwater with a concentration 0.1 kg/L; the other carries 2 L/min of pure water. 5 L/min of water leaves the tank through a drain. At time $t = 0$, the tank contains 100 L of saltwater at the concentration 0.1 kg/L.*
   Set up (but <u>do not solve</u>) a differential equation for the amount of salt in the tank at time t.
   Let $S(t)$ be the amount of salt in the tank at time t.
   The tank has two inputs:

3 L/min at 0.1 kg/L $\longrightarrow$ 0.3 kg/min.
2 L/min at 0 kg/L $\longrightarrow$ 0 kg/min.
The tank has one output. Since the total output of saltwater is the same as the total input of saltwater, the volume of the tank is fixed at 100 L, and the concentration in the tank at time t is $\frac{S(t)}{100}$. So the output rate is $5 \cdot \frac{S(t)}{100}$ kg/min.
We see that the differential equation governing the amount of salt in the tank is

$$S'(t) = 0.3 + 0 - 5 \cdot \frac{S(t)}{100} = 0.3 - \frac{S(t)}{20}.$$