Proof of the Chebyshev inequality (continuous case):

Given: X a real continuous random variables with $E(X) = \mu$, $V(X) = \sigma^2$, real number $\epsilon > 0$.
To show: $P(|X - \mu| \geq \epsilon) \leq \frac{\sigma^2}{\epsilon^2}$.

Then

$$\begin{aligned} \sigma^2 &= V(X) \\ &= \int_{-\infty}^{\infty} (t-\mu)^2 f_X(t)\, dt \\ &\geq \int_{-\infty}^{\mu-\epsilon} (t-\mu)^2 f_X(t)\, dt + \int_{\mu+\epsilon}^{\infty} (t-\mu)^2 f_X(t)\, dt, \end{aligned}$$

where the last line is by restricting the region over which we integrate a positive function. Then this is

$$\geq \int_{-\infty}^{\mu-\epsilon} \epsilon^2 f_X(t)\, dt + \int_{\mu+\epsilon}^{\infty} \epsilon^2 f_X(t)\, dt,$$

since $t \leq \mu - \epsilon \implies \epsilon \leq |t - \mu| \implies \epsilon^2 \leq (t-\mu)^2$. But we rearrange and use the definition of the density function to get

$$\begin{aligned} &= \epsilon^2 \left(\int_{-\infty}^{\mu-\epsilon} f_X(t)\, dt + \int_{\mu+\epsilon}^{\infty} f_X(t)\, dt \right) \\ &= \epsilon^2 P(X \leq \mu - \epsilon \text{ or } X \geq \mu + \epsilon) \\ &= \epsilon^2 P(|X - \mu| \geq \epsilon). \end{aligned}$$

Thus,

$$\sigma^2 \geq \epsilon^2 P(|X - \mu| \geq \epsilon),$$

and dividing through by ϵ^2 gives the desired. □