Math 493

Final Examination – December 20, 2010

Name ________________________________

General Instructions: Please answer the following, showing all your work, and without the aid of books or notes. Total number of points = 40. Write all probabilities involving normal random variables in terms of the standard normal density function.

1. (1 points each) True/False. Please read the statements carefully, as no partial credit will be given.

   (a) ____ If X and Y are (non-independent) continuous random variables, then $E(X+Y) = E(X) + E(Y)$.

   (b) ____ If X and Y are continuous random variables with density functions f_X and f_Y, then $X+Y$ has density function
   $$f_{X+Y}(t) = \int_{-\infty}^{\infty} f_X(t-\heartsuit) f_Y(\heartsuit)\, d\heartsuit.$$

   (c) ____ If X and Y are independent continuous random variables with density functions f_X and f_Y, then $X+2Y$ has density function
   $$f_{X+2Y}(t) = \frac{1}{2}\int_{-\infty}^{\infty} f_X(t-\heartsuit) f_Y(2\heartsuit)\, d\heartsuit.$$

   (d) ____ If $P(A) = 1/2$, and $P(B) = 1/2$, then $P(A \cap B) = 1/4$.

   (e) ____ The expected number of kings in a hand of 13 cards is 1.

   (f) ____ The expected number of kings in a hand of 13 cards, given that there is at least 1, is 2.

   (g) ____ $\binom{11}{0} + \binom{11}{1} + \binom{11}{2} + \binom{11}{3} + \binom{11}{4} + \binom{11}{5} = 1024.$

   (h) ____ If $E(XY) = E(X)E(Y)$, then X and Y are independent.

   (i) ____ If $E(XY) = E(X)E(Y)$, then $V(X+Y) = V(X) + V(Y)$.

   (j) ____ The probability of exactly 3 kings in a hand of 13 cards is $\frac{4}{\binom{52}{13}}$.

2. (a) (5 points) Recall that a *fall* in a permutation $\sigma = i_1 \dots i_n$ of $\{1, \dots, n\}$ is a position k where $i_k > i_{k+1}$; we always consider σ to have a fall at n. Let X be the number of falls of a permutation of $\{1, \dots, n\}$ chosen uniformly at random. Find $E(X)$.

(b) (5 points) Let X be a discrete random variable with expected value. Explain why $E(|X|) < \infty$, and prove that

$$P(|X| \geq a) \leq E(|X|)/a.$$

3. (a) (5 points) Let X be an exponential random variable with parameter λ, and Y be a uniform random variable on $[0, 1]$. If X and Y are independent, find the density function $g(t)$ of $X + Y$.
(Please evaluate any integrals.)

(b) (5 points) A certain radioactive element (rabbitonium) averages a radioactive emission every second. Let A be the time elapsed until you have 400 radioactive emissions. Estimate the probability that $360 \leq A \leq 440$. Explain carefully what (if any) results from class you are using.

4. (a) (3 points) Let X be a continuous random variable with a density function. Find the cumulative distribution function and density function for $|X|$.

(b) (2 points) Let X be a continuous random variable with expected value. Show that

$$E(|X|) = \int_{-\infty}^{\infty} |t| f_X(t)\, dt.$$

(Notice that $\varphi(t) = |t|$ is neither increasing or decreasing!)

(c) (4 points) Let X and Y be continuous independent random variables. Show that $|X|$ and Y are independent. (You may use a problem from a previous exam.)

(d) (1 point) Conclude from (c) that $|X|$ and $|Y|$ are also independent.