Math 132
Bounding error terms – February 17, 2012

1. Working with inequalities

Three useful rules for working with inequalities are:

(1) $|a \cdot b| = |a| \cdot |b|$
(2) $|a + b| \leq |a| + |b|$ (the *triangle inequality*)
(3) $|a - b| \leq |a| + |b|$

Rule (3) may seem surprising. Let us see how to prove it. We calculate:

$$|a - b| = |a + (-b)| \leq |a| + |-b| = |a| + |b|.$$

2. Bounds

Definition 1.
An *upper bound* for a function f is a number U so that: for all x, we have $f(x) \leq U$.
A *lower bound* for a function f is a number L so that: for all x, we have that $f(x) \geq L$.

A *bound in absolute value*, which is what we will usually refer to as just a *bound*, is a number M so that $|f(x)| \leq M$ for all x.

Notice that if M is a bound in absolute value for f, then $-M$ and M are lower and upper bounds for f, and conversely that if L and U are lower and upper bounds, then $\max(|L|, |U|)$ is a bound for f in absolute value.

We'll usually be interested in bounds in absolute value, since they are convenient and quick to work with.

Definition 2. We say f has an *upper bound* U on the interval $[a, b]$ if: for all x on $[a, b]$, we have $f(x) \leq U$. Similarly for lower bounds and bounds in absolute value.

Example 3. Some bounds:

(1) $|\sin x| \leq 1$ for all x. Thus, 1 is a bound (in absolute value) for $\sin x$. So is 2, and 3, and 3.17, but not 0.98.
(2) $|x^3| \leq 27$ on the interval $[-3, 1]$.
(3) $|e^x| \leq e^2$ on the interval $[-5, 2]$. We could also say that $|e^x| \leq 9$ on the interval $[-5, 2]$, since $9 > e^2$.

2.1. Geometric interpretation.

Geometric interpretation. Geometrically, an upper bound is a horizontal line that the graph of the function does not go above. Similarly, a lower bound is a horizontal line that the graph does not go below. A bound in absolute value 'traps' the graph of the function in a band between the horizontal lines $y = -M$ and $y = M$.

3. Example

We find a bound for $12 \sin x^2 - x \cos x$ on the interval $[-3, 2]$. Using the rules from Part 1, we break it apart:

$$\begin{aligned} |12 \sin x^2 - x \cos x| &\leq |12 \sin x^2| + |x \cos x| \\ &= 12|\sin x^2| + |x| \cdot |\cos x|. \end{aligned}$$

Since $|x| \leq 3$ on the interval $[-3, 2]$, and since $|\sin *| \leq 1$ and $|\cos *| \leq 1$, we get that

$$|12 \sin x^2 - x \cos x| \leq 12 \cdot 1 + 3 \cdot 1 = 15 \text{ (on the interval } [-3, 2]).$$

Note: this is not the least possible bound. However, for many applications it is good enough.

4. Error bounds for numerical integration

To find an error bound for the Trapezoid Rule or Simpson's rule, we first need to find an upper bound for $|f''(x)|$ or $\left|f^{(4)}(x)\right|$ (respectively) on the interval in question.

Example 4. Find a number of subintervals so that the error of the Trapezoid Rule applied to $\int_0^1 \sin x^2 \, dx$ is at most 0.1.

Solution: We calculate derivatives: $f(x) = \sin(x^2)$, so $f'(x) = \cos(x^2) \cdot 2x$ and

$$f''(x) = -\sin(x^2) \cdot 4x^2 + \cos(x^2) \cdot 2 = -4x^2 \sin(x^2) + 2 \cos(x^2).$$

So

$$\begin{aligned} |f''(x)| &\leq \left|4x^2 \sin(x^2)\right| + \left|2 \cos(x^2)\right| \\ &= 4 \cdot \left|x^2\right| \cdot \left|\sin(x^2)\right| + 2 \left|\cos(x^2)\right|. \end{aligned}$$

As previously discussed, $|\sin(\text{anything})| \leq 1$, and similarly for cos. Since we are integrating on the interval $[0, 1]$, we have $0 \leq x \leq 1$ and in particular $|x^2| \leq 1$. Thus,

$$|f''(x)| \leq 4 \cdot 1 \cdot 1 + 2 \cdot 1 = 6.$$

Plugging into the error formula for the Trapezoid Rule, we get that

$$|E_T| \leq \frac{M \cdot (b-a)^3}{12n^2} = \frac{6 \cdot (1-0)^3}{12n^2} = \frac{1}{2n^2}.$$

So to have $|E_T| \leq 0.1$, it suffices to set $\frac{1}{2n^2} \leq 0.1$. Solving this, we get $2n^2 \geq 10$, i.e., $n \geq \sqrt{5}$. Since $3^2 > 5$, $n = 3$ will give already give the desired accuracy!

Exercise 5. How many subintervals do you want in order to force the error of the Trapezoid Rule to be less than 0.01 for $\int_0^2 \sin x^2 \, dx$?

Exercise 6. How many subintervals do you want in order to force the error of the Trapezoid Rule to be less than 0.01 for each of the following integrals?

$$\int_{-2}^{3} e^{x^2} \, dx, \qquad \int_0^{\sqrt{\pi}} \cos(x^2) \, dx$$

Exercise 7. For the integrals from Exercise 6, how many subintervals to force the error of Simpson's Rule to be less than 0.01?