Lifting Measures to R^1, R^k, and R^∞

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1. Introduction. The purpose here is to prove

$$\int_Y f(u)\,\mu_X(du) \;=\; \int_\Omega f\big(X(\omega)\big)P(d\omega) \;=\; E\Big(f(X)\Big) \tag{1}$$

where

$$\mu_X(B) \;=\; P(X\in B), \quad B\in\mathcal{A}$$

In (1), $X:\Omega\to Y$ is a measurable mapping from a probability space $(\Omega,\mathcal{F},P)$ to a measurable space $(Y,\mathcal{A})$ (that is, X is a Y-valued random variable) and $f(u)\ge 0$ is an $\mathcal{A}$-measurable function on Y. The measure $\mu_X(B)$ is the measure on $(Y,\mathcal{A})$ that is "induced" or "lifted" from P on $(\Omega,\mathcal{F})$ by the mapping $X:\Omega\to Y$.

The motivating examples for (1) are (i) $(Y,\mathcal{A})=(R,\mathcal{B}(R))$ for real-valued random variables $X(\omega)$, (ii) $(Y,\mathcal{A})=(R^k,\mathcal{B}(R^k))$ for vector-valued random variables $X=(X_1,X_2,\dots,X_k)$, and (iii) $(Y,\mathcal{A})=(R^\infty,\mathcal{B}(R^\infty))$ for infinite-sequence-valued random variables $X=(X_1,X_2,\dots,X_n\dots)$.

For $(Y,\mathcal{A})=(R^k,\mathcal{B}(R^k))$, the relation (1) takes the form

$$\int_{R^k} f(u_1,\dots,u_k)\,\mu_X(du) \;=\; E\Big(f(X_1,\dots,X_k)\Big)$$

where $f(u_1,\dots,u_k)\ge 0$ is an arbitrary Borel function on R^k and $\mu_X(du)$ is a generalized Lebesgue-Steiltjes measure on R^k.

A standard definition for a mapping $X:\Omega\to Y$ to be $\mathcal{A}$-measurable is that

$$\{\,\omega : X(\omega)\in A\,\}\in\mathcal{F} \tag{2}$$

for all $A\in\Gamma$ where $\Gamma\subseteq\mathcal{A}$ is a collection of sets with $\mathcal{B}(\Gamma)=\mathcal{A}$. Here $\mathcal{B}(\Gamma)$ is the smallest σ-algebra of subsets of Y that contains Γ.

If $(Y,\mathcal{A})=(R,\mathcal{B}(R))$, then Γ is the set of semi-infinite intervals $A=(-\infty,\lambda]$, so that $\{\,\omega : X(\omega)\in A\,\}=\{\,\omega : X(\omega)\le\lambda\,\}$ and $\mu_X(A)=P(X\le\lambda)$ is the distribution function. If $(Y,\mathcal{A})=(R^k,\mathcal{B}(R^k))$, then Γ is the set of octants $A=\prod_{j=1}^k(-\infty,\lambda_j]$ and $\mu_X(A)$ is the k-dimensional distribution function. In both cases $\mathcal{B}(\Gamma)$ are the Borel sets in R or R^k, respectively.

Condition (2) for $A\in\Gamma$ can be written

$$X^{-1}:\Gamma\to\mathcal{F} \tag{3}$$

where $X^{-1}(E)=\{\,\omega : X(\omega)\in E\,\}$. The general lifting theorem is

Theorem. Suppose that $X : \Omega \to Y$ satisfies (3) for some class $\Gamma \subseteq \mathcal{A}$ with $\mathcal{B}(\Gamma) = \mathcal{A}$. Then $X^{-1} : \mathcal{A} \to \mathcal{F}$ and the set function

$$\mu_X(B) \ = \ PX^{-1}(B) \ = \ P\big(\{\, \omega : X(\omega) \in B \,\}\big) \tag{4}$$

is a probability measure on $\mathcal{A}$. Moreover,

$$\int_Y f(u)\mu_X(du) \ = \ \int_\Omega f\big(X(\omega)\big)P(d\omega) \ = \ E\big(f(X)\big) \tag{5}$$

for all $\mathcal{A}$-measurable functions $f(u) \ge 0$.

Proof. (i) Let

$$\mathcal{Q} = \{\, B \subseteq Y : X^{-1}(B) \in \mathcal{F} \,\}$$

Then $\Gamma \subseteq \mathcal{Q}$ by (3). Also (i) $\phi \in \mathcal{Q}$ for the empty set ϕ since $X^{-1}(\phi) = \phi$, (ii) $B \in \mathcal{Q}$ implies $B^c \in \mathcal{Q}$ since $X^{-1}(B^c) = \big(X^{-1}(B)\big)^c \in \mathcal{F}$, and (iii) $B_j \in \mathcal{Q}$ implies $B = \bigcup_{j=1}^\infty B_j \in \mathcal{Q}$ since $X^{-1}(B) = \bigcup_{j=1}^\infty X^{-1}(B_j) \in \mathcal{F}$. It follows that $\mathcal{Q}$ is a σ-algebra containing Γ. Thus $\Gamma \subseteq \mathcal{B}(\Gamma) = \mathcal{A} \subseteq \mathcal{Q}$, which implies $X^{-1} : \mathcal{A} \to \mathcal{F}$.

(ii) I claim that $\mu_X(B) = PX^{-1}(B)$ is a probability measure on $\mathcal{A}$. Clearly $\mu_X(\phi) = 0$ for the empty set ϕ. Assume $B_j \in \mathcal{A}$ are disjoint and $B = \bigcup_{j=1}^\infty B_j$. Then the $X^{-1}(B_j)$ are disjoint with $X^{-1}(B) = \bigcup_{j=1}^\infty X^{-1}(B_j)$. Since $\mu_X = PX^{-1}$, this implies $\mu_X(B) = \sum_{j=1}^\infty \mu_X(B_j)$ and $\mu_X = PX^{-1}$ is a measure on $\mathcal{A}$. It is a probability measure since $\mu_X(Y) = P(\Omega) = 1$.

(iii) For simple functions $\phi(u) = \sum_{j=1}^n c_j I_{B_j}(x)$, the integral is

$$\begin{aligned} \int_Y \phi(u)\mu_X(du) \ &= \ \sum_{j=1}^n c_j \mu_X(B_j) \ = \ \sum_{i=1}^n c_j PX^{-1}(B_j) \\ &= \ \sum_{j=1}^n c_j P(X \in B_j) \ = \ E\Big(\sum_{j=1}^n c_j I_{\{X \in B_j\}}\Big) \\ &= \ E\Big(\sum_{j=1}^n c_j I_{B_j}(X)\Big) \ = \ E\Big(\phi\big(X(\omega)\big)\Big) \end{aligned} \tag{6}$$

For any $\mathcal{A}$-measurable $f : Y \to R^1$ with $f(u) \ge 0$, there exist simple functions $\phi_n(u)$ such that $0 \le \phi_n(u) \uparrow f(u)$ for all u. The Monotone Convergence Theorem in (6) then implies (5), which completes the proof of the theorem.