Measures on Semi-Rings in R^1 and R^k

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1. Introduction. The purpose here is to provide an efficient way of deriving Borel measures (in particular Lebesgue-Steiltjes measures in R^1 and R^k) using semi-rings of subsets of a set X. We feel that this is a more efficient and more heuristic approach that using algebras of subsets of X, even though using algebras may provide shorter proofs if certain combinatorial lemmas are viewed as obvious.

2. Semi-rings of Sets. In general, a *semi-ring* of subsets of a set X is a collection Γ of subsets of X such that

(i) $\phi \in \Gamma$
(ii) $A, B \in \Gamma$ implies $A \cap B \in \Gamma$
(iii) For any $A, B \in \Gamma$, there exists an integer m and disjoint sets $C_1, \ldots, C_m \in \Gamma$ such that $A - B = \bigcup_{j=1}^m C_j$.

Examples: (1) $P(X) = 2^X$, the set of all subsets of X.

(2) For $X = R^1$, the set Γ of all *cells* or *h-intervals* $(a, b]$ for $-\infty < a \le b < \infty$. Another example is the slightly larger collection Γ_1 of cells with $-\infty \le a \le b \le \infty$. Note that the condition $a = b$ allows $\phi \in \Gamma$.

(3) For $X = R^k$, the set Γ of all cells $\prod_{j=1}^k (a_j, b_j]$ where $\prod$ denotes the Cartesian product and $-\infty < a_j \le b_j < \infty$. As in Example (2), we can also allow $a_j = -\infty$ and $b_j = \infty$.

(4) Any σ-algebra $\mathcal{M}$ of subsets of X. Recall that $\mathcal{M}$ is a σ-algebra of subsets of X if

(i) $\phi \in \mathcal{M}$
(ii) $A \in \mathcal{M}$ implies $A^c \in \mathcal{M}$
(iii) If $A_j \in \mathcal{M}$ for $1 \le j < \infty$, then $A = \bigcup_{j=1}^\infty A_j \in \mathcal{M}$.

Exercise: Verify that Examples (2) and (3) are semi-rings, and that we can take $m \le 2$ in part (iii) for Example (2) and $m \le 2^k$ in Example (3).

Definition: A set function $\mu(A)$ is a *premeasure* or *countably-additive measure* on a semi-ring Γ if $\mu : \Gamma \to [0, \infty]$ is a function such that

(i) $\mu(\phi) = 0$

(ii) If $A_k \in \Gamma$ are disjoint for $1 \le k < \infty$, and if $A = \bigcup_{k=1}^{\infty} A_k \in \Gamma$, then $\mu(A) = \sum_{k=1}^{\infty} \mu(A_k)$.

A premeasure μ_0 is *σ-finite* if $X = \bigcup_{j=1}^{\infty} X_j$ where $X_j \in \Gamma$ and $\mu_0(X_j) < \infty$.

Notes: (a) In particular, $\mu(A) = \infty$ for $A \in \Gamma$ is allowed.

(b) $\mu(A)$ is also *finitely-additive* on Γ. That is, if $A, A_j \in \Gamma$, A_j is disjoint, and $A = \bigcup_{j=1}^{n} A_j$ satisfies $A \in \Gamma$, then $\mu(A) = \sum_{j=1}^{n} \mu(A_j)$. This is because we can take $A_j = \phi$ for $j > n$ in property (ii) above. If $\mu(A)$ is only finitely additive; that is, if (ii) is only guaranteed if $A_j = \phi$ for $j > n$ for some finite n, then we call $\mu(A)$ a *finitely-additive premeasure* on Γ.

(c) If $A, B \in \Gamma$ and $A \subseteq B$, then $\mu(A) \le \mu(B)$ by property (iii) of the definition of a semi-ring, property (ii) of the definition of a premeasure, and the property that $\mu(C) \ge 0$ for $C \in \Gamma$. Thus a premeasure (or finitely-additive premeasure) on a semi-ring is automatically monotone.

Definition: $\mu(A)$ is a *measure* on a σ-algebra $\mathcal{M}$ if $\mu : \mathcal{M} \to [0, \infty]$ satisfies

(i) $\mu(\phi) = 0$

(ii) If $A_k \in \mathcal{M}$ are disjoint for $1 \le k < \infty$ and $A = \bigcup_{k=1}^{\infty} A_k$ (which is automatically in $\mathcal{M}$), then $\mu(A) = \sum_{k=1}^{\infty} \mu(A_k)$.

The following three lemmas are useful for working with semi-rings.

Lemma 2.1. Assume sets $A, A_1, \ldots, A_n \in \Gamma$ for a semi-ring Γ. Then there exists $m < \infty$ and disjoint sets $D_1, D_2, \ldots, D_m \in \Gamma$ such that

$$A - \bigcup_{j=1}^{n} A_j \;=\; A - A_1 - A_2 - \cdots - A_n \;=\; \bigcup_{k=1}^{m} D_k \tag{2.1}$$

Proof. By condition (iii) for semi-rings, $A - A_1 = \bigcup_{j=1}^{m} C_j$ for disjoint $C_j \in \Gamma$. Then $A - A_1 - A_2 = \bigcup_{j=1}^{m} C_j - A_2 = \bigcup_{j=1}^{m}(C_j - A_2) = \bigcup_{j=1}^{m} \bigcup_{k=1}^{n_j} D_{jk}$ where $D_{jk} \in \Gamma$ are disjoint for fixed j with $\bigcup_{k=1}^{n_j} D_{jk} = C_j - A_2$. Since the C_j are disjoint with $D_{jk} \subseteq C_j$, the D_{jk} are disjoint for all j, k. Thus we can write $A - A_1 - A_2 = \bigcup_{k=1}^{M} \widetilde{D}_k$ for disjoint $\widetilde{D}_k \in \Gamma$ and $M \le n_1 + \ldots + n_m$. Lemma 2.1 for all n follows by induction on n.

Exercise: Show that we can take $m \le 2^n$ for the semi-ring of cells Γ in Example (2). For cells in R^k (Example (3)), we can take $m \le 2^{nk}$.

Lemma 2.2. Let $\mu(A)$ be a finitely-additive premeasure on a semi-ring Γ. Assume $A, A_1, \ldots, A_n \in \Gamma$ are such that $A_1, \ldots, A_n$ are disjoint and $\bigcup_{j=1}^n A_j \subseteq A$. Then

$$\sum_{j=1}^n \mu(A_j) \;\le\; \mu(A) \tag{2.2}$$

Proof. By Lemma 2.1, $A-\bigcup_{j=1}^n A_j = \sum_{k=1}^m D_k$ where $D_k \in \Gamma$ are disjoint, also disjoint from $A_1, \ldots, A_n$. Thus $\{ A_1, \ldots, A_n, D_1, \ldots, D_m \}$ are disjoint and by finite additivity

$$\mu(A) \;=\; \sum_{j=1}^n \mu(A_j) \;+\; \sum_{k=1}^m \mu(D_k) \;\;\ge\;\; \sum_{j=1}^n \mu(A_j)$$

since $\mu(D_k) \ge 0$.

Lemma 2.3. Let $\mu(A)$ be a finitely-additive premeasure on a semi-ring Γ. Assume $A, A_1, \ldots, A_n \in \Gamma$ are such that $A \subseteq \bigcup_{j=1}^n A_j$. Then

$$\mu(A) \;\le\; \sum_{j=1}^n \mu(A_j) \tag{2.3}$$

Proof. Since $A \subseteq \bigcup_{j=1}^n A_j$,

$$A \;=\; A \cap \bigcup_{j=1}^n A_j \;=\; \bigcup_{j=1}^n (A\cap A_j) \;=\; \bigcup_{j=1}^n \widetilde{A}_j, \quad \widetilde{A}_j = (A\cap A_j) - \bigcup_{k=1}^{j-1} (A\cap A_k)$$

Each $A \cap A_k \in \Gamma$ by condition (ii) of the definition of a semi-ring. The sets $\widetilde{A}_j$ are disjoint, but are not necessarily in Γ. By Lemma 2.1, each $\widetilde{A}_j = \bigcup_{k=1}^{n_j} D_{jk}$ where $D_{jk} \in \Gamma$ are disjoint for fixed j. Since the $\widetilde{A}_j$ are disjoint, the sets $D_{jk} \in \Gamma$ are disjoint for all j, k. Since μ is finitely additive,

$$\mu(A) \;=\; \sum_{j=1}^n \sum_{k=1}^{n_j} \mu(D_{ij}) \;\le\; \sum_{j=1}^n \mu(A_j)$$

by Lemma 2.2 since $\bigcup_{k=1}^{n_j} D_{jk} = \widetilde{A}_j \subseteq A_j$.

3. Semi-rings and Outer Measures. An *outer measure* on a set X is a function $\mu^* : P(X) \to [0,\infty]$ where $P(X)$ is the set of all subsets $E \subseteq X$ such that μ^* satisfies

(i) $\mu^*(\phi) = 0$

(ii) $E \subseteq F \subseteq X$ implies $\mu^*(E) \le \mu^*(F)$

(iii) If $E_j \subseteq X$ for $1 \le j < \infty$ and $E = \bigcup_{j=1}^{\infty} E_j$, then $\mu^*(E) \le \sum_{j=1}^{\infty} \mu^*(E_j)$.

Note that outer measures are defined for all subsets E of a set X rather than on a semi-ring or σ-algebra.

Definition: A set $A \subseteq X$ is μ^**-measurable* if

$$\mu^*(E) \ = \ \mu^*(E \cap A) \ + \ \mu^*(E \cap A^c) \tag{3.1}$$

for all subsets $E \subseteq X$. Define $\mathcal{M}(\mu^*)$ as the set of all μ^*-measurable subsets $A \subseteq X$.

In particular, $A = \phi \in \mathcal{M}(\mu^*)$ since (3.1) holds for all $E \subseteq X$. Similary, $A \in \mathcal{M}(\mu^*)$ implies $A^c \in \mathcal{M}(\mu^*)$, which are two of the three properties required for a σ-algebra. More generally:

Theorem 3.1 (Carathéodory) Let μ^* be an arbitrary outer measure on a set X. Then

(i) $\mathcal{M}(\mu^*)$ is a σ-algebra of subsets of X

(ii) μ^* is a (countably-additive) measure on $\mathcal{M}(\mu^*)$.

Proof. See Folland (1999) in the references, or any textbook on measure theory. (This proof does not use semi-rings or algebras of sets.)

Notes: (1) Theorem 3.1 does not guarantee that the σ-algebra is very large or very interesting. Problem 4 on Homework 1 of Math 5051 (Fall 2009) gives an example of an outer measure μ^* on $X = [0,1]$ with $\mu^*(E) > 0$ for all nonempty $E \subseteq [0,1]$ but $\mathcal{M}(\mu^*) = \{\, \phi, X \,\}$.

(2) Let $\mathcal{E} \subseteq P(X)$ be an arbitrary collection of subsets of a set X and let $\mu_0(A)$ be an arbitrary nonnegative function on $\mathcal{E}$. Then

$$\mu^*(E) = \inf \left\{ \sum_{j=1}^{\infty} \mu_0(A) \ : \ E \subseteq \bigcup_{j=1}^{\infty} A_j, \ A_j \in \mathcal{E} \right\} \tag{3.2}$$

defines an outer measure on X. We define $\mu^*(E)$ with the convention that the infimum of the empty set is ∞. That is, if E cannot be covered by a sequence of sets $A_j \in \mathcal{E}$ as in (3.2), then $\mu^*(E) = \infty$. (*Proof*: See Folland (1999).)

Definition: H is a *(μ^*)-null set* if $H \subseteq X$ and $\mu^*(H) = 0$. If H is a μ^*-null set, then $H \in \mathcal{M}(\mu^*)$. That is, $\mathcal{M}(\mu^*)$ contains all null sets for μ_0. (*Proof*: If $\mu^*(H) = 0$, then

$$\mu^*(E) \ \le\ \mu^*(E \cap H) + \mu^*(E \cap H^c) \ \le\ \mu^*(E \cap H^c) \ \le\ \mu^*(E)$$

and (3.1) holds for all $E \subseteq X$. Hence $H \in \mathcal{M}(\mu^*)$.)

The next result shows how to extend an arbitrary premeasure on a semi-ring to a measure on a σ-algebra.

Theorem 3.2 (Carathéodory) Let μ_0 be a (countably-additive) premeasure on a semi-ring Γ of subsets of a set X. Define $\mu^*(E)$ by (3.2) for $\mathcal{E} = \Gamma$. Then

(i) $\mu^*(A) = \mu_0(A)$ for all $A \in \Gamma$

(ii) $\Gamma \subseteq \mathcal{M}(\mu^*)$.

Notes: (1) For $\mu^*(E)$ as in Theorem 3.2, if we define $\mu(A) = \mu^*(A)$ for $A \in \mathcal{M}(\mu^*)$, then μ is a measure on both $\mathcal{M}(\mu^*)$ and on the smallest σ-algebra $\mathcal{M}(\Gamma)$ containing Γ.

(2) Under the conditions of Theorem 3.2, if μ_0 is σ-finite on X, then every $E \in \mathcal{M}(\mu^*)$ can be written $E = B - H$ where $B \in \mathcal{M}(\Gamma)$ and $\mu^*(H) = 0$. That is, $\mathcal{M}(\mu^*)$ differs from $\mathcal{M}(\Gamma)$ only by null sets. (See Problem 2 on Homework 2 for Math 5051, Fall 2009.)

Proof of Theorem 3.2 (Carathéodory). (i) We first show that $\mu^*(A) = \mu_0(A)$ for any $A \in \Gamma$. Since A is a covering of itself, $\mu^*(A) \le \mu_0(A)$. Thus is is sufficient to prove $\mu_0(A) \le \mu^*(A)$.

(*Remark:* Problem 5 of Homework 2 in Math 5051 (Fall 2009) gives an example of an outer measure defined by (3.2) with $\mu_0(A) > 0$ for every nonempty $A \in \Gamma$ but $\mu^*(E) = 0$ for all sets $E \subseteq X$. Thus some argument is required.)

Given $A \in \Gamma$ with $\mu^*(A) < \infty$ (otherwise $\mu_0(A) \le \mu^*(A)$ is trivial), choose $A_i \in \Gamma$ such that

$$A \ \subseteq\ \bigcup_{j=1}^{\infty} A_j, \qquad \mu^*(A) \ \le\ \sum_{j=1}^{\infty} \mu_0(A_j) \ \le\ \mu^*(A) + \epsilon$$

As in the proof of Lemma 2.3, we can find disjoint $B_k \in \Gamma$ such that

$$A \ \subseteq\ \bigcup_{j=1}^{\infty} A_j \ =\ \bigcup_{k=1}^{\infty} B_k, \qquad \mu^*(A) \ \le\ \sum_{k=1}^{\infty} \mu_0(B_k) \ \le\ \sum_{j=1}^{\infty} \mu_0(A_j)$$

Then $A = \bigcup_{k=1}^{\infty}(A \cap B_k)$ for disjoint sets $A \cap B_k \in \Gamma$. Thus

$$\mu_0(A) \;=\; \sum_{k=1}^{\infty} \mu_0(A \cap B_k) \;\le\; \sum_{k=1}^{\infty} \mu_0(B_k) \;\le\; \sum_{j=1}^{\infty} \mu_0(A_j) \;\le\; \mu^*(A) + \epsilon$$

since $\mu_0(A) \le \mu_0(B)$ if $A \subseteq B$, $A, B \in \Gamma$. This implies $\mu_0(A) \le \mu^*(A)$ and hence $\mu_0(A) = \mu^*(A)$.

(ii) We next show that any $A \in \Gamma$ satisfies $A \in \mathcal{M}(\mu^*)$. Since μ^* is subadditive (that is, property (iii) of the definition of outer measure), it is sufficient to prove

$$\mu^*(E) \;\ge\; \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

for all subsets $E \subseteq X$. Choose $A_j \in \Gamma$ such that

$$E \;\subseteq\; \bigcup_{j=1}^{\infty} A_j, \qquad \mu^*(E) \le \sum_{j=1}^{\infty} \mu_0(A_j) \le \mu^*(E) + \epsilon$$

By property (iii) of the definition of a semi-ring

$$E \cap A \;\subseteq\; \bigcup_{j=1}^{\infty} (A \cap A_j), \qquad E \cap A^c \;\subseteq\; \bigcup_{j=1}^{\infty} (A_j - A) \;=\; \bigcup_{j=1}^{\infty} \bigcup_{k=1}^{n_j} D_{jk}$$

where $A_j - A = \bigcup_{k=1}^{n_j} D_{jk}$ for disjoint $D_{jk} \in \Gamma$. Thus

$$\begin{aligned}\mu^*(E \cap A) + \mu^*(E \cap A^c) \;&\le\; \sum_{j=1}^{\infty} \mu_0(A \cap A_j) \;+\; \sum_{j=1}^{\infty} \sum_{k=1}^{n_j} \mu_0(D_{jk}) \\ &= \sum_{j=1}^{\infty} \Bigl(\mu_0(A \cap A_j) + \sum_{k=1}^{n_j} \mu_0(D_{jk}) \Bigr) \;=\; \sum_{j=1}^{\infty} \mu_0(A_j) \;\le\; \mu^*(E) + \epsilon\end{aligned}$$

since $A_j = (A \cap A_j) \cup (A_j - A)$ and μ_0 is finitely additive. Thus $\mu^*(E) \ge \mu^*(E \cap A) + \mu^*(E \cap A^c)$ for all $E \subseteq X$, which completes the proof of the theorem.

An important corollary of Theorem 3.2 is

Theorem 3.3 (Fréchet) Let μ and ν be two measures on a σ-algebra $\mathcal{M}$ on a set X. Assume $\Gamma \subseteq \mathcal{M}$ for a semi-ring Γ, that $\mu(A) = \nu(A)$ for every $A \in \Gamma$, and that μ an ν are both σ-finite on Γ. Then $\mu(E) = \nu(E)$ for all $E \in \mathcal{M}$.

Proof. This is essentially Theorem 1.14 in Folland (1999).

Define $\mu^*(E)$ and $\nu^*(E)$ by (3.2) for $\mathcal{E} = \Gamma$. Then $\mu^*(E) = \nu^*(E)$ for all $E \subseteq X$ since $\mu(A) = \nu(A)$ for $A \in \Gamma$. Since measures are countably subadditive, $\mu(E) \le \mu^*(E)$ and $\nu(E) \le \nu^*(E)$ for all $E \in \mathcal{M}$. If we can show that $\mu(E) = \mu^*(E)$ for any $E \in \mathcal{M}$ and σ-finite premeasure μ on Γ, then we could conclude $\mu(E) = \mu^*(E) = \nu^*(E) = \nu(E)$ for all $E \in \mathcal{M}(\Gamma)$ and we would be done.

First, assume $\mu^*(E) < \infty$. As in the proofs of Lemma 2.3 and Theorem 3.3, there exist disjoint sets $A_j \in \Gamma$ such that

$$E \subseteq A = \bigcup_{j=1}^{\infty} A_j, \qquad \mu^*(E) \le \sum_{j=1}^{\infty} \mu(A_j) = \mu(A) \le \mu^*(E) + \epsilon$$

Since the A_j are disjoint and $\mu^*(A_j) = \mu(A_j)$ by Theorem 3.3, $\mu^*(A) = \mu(A)$. Since $E \subseteq A$ and $\mu^*(A) \le \mu^*(E) + \epsilon$, $\mu^*(A - E) \le \epsilon$. Thus also $\mu(A - E) \le \mu^*(A - E) \le \epsilon$ and

$$\mu^*(E) \le \mu^*(A) = \mu(A) = \mu(E) + \mu(A - E) \le \mu(E) + \epsilon$$

Thus $\mu^*(E) \le \mu(E)$ and hence $\mu^*(E) = \mu(E)$ for $\mu^*(E) < \infty$.

Since μ is σ-finite, $X = \bigcup_{j=1}^{\infty} X_j$ where $X_j \in \Gamma$, $\mu(X_j) < \infty$, and X_j are disjoint. Then for all $E \in \mathcal{M}(\Gamma)$

$$\mu(E) = \sum_{j=1}^{\infty} \mu(E \cap X_j) = \sum_{j=1}^{\infty} \mu^*(E \cap X_j) = \mu^*(E)$$

and $\mu = \mu^*$ on $\mathcal{M}(\Gamma)$, which was to be proven.

4. Lebesgue-Stieltjes Measures in $\boldsymbol{R^1}$. Let $F(x)$ be an increasing real-valued right-continuous function on R^1. Let Γ be the semi-ring

$$\Gamma = \{ (a, b] : -\infty < a \le b < \infty \} \tag{4.1}$$

Define μ_F on Γ by

$$\mu_F(\,(a, b]\,) = F(b) - F(a) \tag{4.2}$$

Then

Theorem 4.1. μ_F in (4.2) is a premeasure on the semi-ring Γ. In particular, μ_F in (4.2) extends to a unique Borel measure on R^1.

Proof. Since $F(x)$ is real-valued, $\mu_F(\,(n, n+1]\,) < \infty$ and μ_F is σ-finite. Once we prove that μ_F is a premeasure on Γ, it follows from Theorems 3.3 and 3.4 that μ_F has a unique extension as a Borel measure on $\mathcal{M}(R^1)$. This will be the Lebesgue-Stieltjes measure on R^1 corresponding to $F(x)$. In particular, it is sufficient to prove that μ_F is a premeasure.

Assume $A = (a, b]$ and $A_j = (a_j, b_j]$ satisfy

$$(a, b] \;=\; \bigcup_{j=1}^{\infty} (a_j, b_j] \qquad \text{where } (a_j, b_j] \text{ are disjoint}$$

By Lemma 2.2, $\sum_{j=1}^{n} (a_j, b_j] \subseteq (a, b]$ implies

$$\sum_{j=1}^{n} \big(F(b_j) - F(a_j)\big) \;=\; \sum_{j=1}^{n} \mu_F(A_j) \;\le\; \mu_F(A) \;=\; F(b) - F(a)$$

for all n. Hence

$$\sum_{j=1}^{\infty} \big(F(b_j) - F(a_j)\big) \;\le\; F(b) - F(a) \tag{4.3}$$

Thus it is sufficient to prove

$$F(b) - F(a) \;\le\; \sum_{j=1}^{\infty} \big(F(b_j) - F(a_j)\big) \tag{4.4}$$

For any $\epsilon > 0$, there exist $\delta > 0$ and $\delta_j > 0$ such that

$$F(a + \delta) - F(a) < \epsilon \qquad \text{and} \qquad F(b_j + \delta_j) - F(b_j) < \epsilon/2^j \tag{4.5}$$

for all $j \ge 1$. Then

$$[a + \delta, b] \;\subseteq\; (a, b] \;=\; \bigcup_{j=1}^{\infty} (a_j, b_j] \;\subseteq\; \bigcup_{j=1}^{\infty} (a_j, b_j + \delta_j)$$

Since $[a + \delta, b]$ is compact, it follows that

$$[a + \delta, b] \;\subseteq\; \bigcup_{j=1}^{n} (a_j, b_j + \delta_j)$$

for some $n < \infty$. By Lemma 2.3 and (4.5)

$$\begin{aligned} \mu_F(\,(a,b]\,) \;&=\; F(b) - F(a) \;\le\; \big(F(b) - F(a+\delta)\big) \;+\; \epsilon \\ &\le\; \sum_{j=1}^{n} \big(F(b_j+\delta_j) - F(a_j)\big) + \epsilon \\ &\le\; \sum_{j=1}^{n} \big(F(b_j) - F(a_j)\big) \;+\; \sum_{j=1}^{n} \epsilon/2^j + \epsilon \\ &\le\; \sum_{j=1}^{\infty} \big(F(b_j) - F(a_j)\big) \;+\; 2\epsilon \end{aligned}$$

Since this holds for all $\epsilon > 0$, we conclude (4.4) and hence that μ_F is a premeasure on Γ in (4.1).

5. Lebesgue-Stieltjes Measures in R^k. Let Γ be the semi-ring

$$\Gamma \;=\; \left\{ C \,:\, C = \prod_{j=1}^{k} (a_j, b_j] \quad \text{for } -\infty < a_j \le b_j < \infty, \quad 1 \le j \le k \right\} \tag{5.1}$$

where $\prod_{j=1}^{k}(a_j, b_j]$ means Cartesian product. As in Section 2 (Example 3), Γ is a semi-ring of subsets of R^k. If μ is a Borel measure on R^k and $\mu(R^k) < \infty$, an analog for R^k of the increasing function $F(x)$ in Section 4 is

$$F(x_1, x_2, \ldots, x_k) \;=\; \mu\bigg(\prod_{j=1}^{k} (-\infty, x_j] \bigg) \tag{5.2}$$

We now want the analog of $\mu\big(\,(a,b]\,\big) = F(b) - F(a)$ in Section 4. Consider the special case of the *product measure*

$$\mu = \mu_1 \otimes \mu_2 \otimes \ldots \otimes \mu_k$$

for one-dimensional measures μ_j. This means

$$F(x_1, x_2, \ldots, x_k) \;=\; F_1(x_1) F_2(x_2) \ldots F_k(x_k)$$

where $F_j(x) = \mu_j\big(\,(-\infty, x]\,\big)$ and

$$\mu\bigg(\prod_{j=1}^{k} (a_j, b_j] \bigg) \;=\; \prod_{j=1}^{k} \big(F_j(b_j) - F_j(a_j)\big) \tag{5.3}$$

In particular, if $C = (a,b] \times (c,d] \subseteq R^2$, then

$$\begin{aligned} \mu\big((a,b] \times (c,d] \,\big) \;&=\; \big(F_1(b) - F_1(a)\big)\big(F_2(d) - F_2(c)\big) \\ &=\; F_1(b)F_2(d) - F_1(a)F_2(d) - F_1(b)F_2(c) + F_1(a)F_2(c) \end{aligned}$$

If $F(x_1, x_2)$ in (5.2) is not a product, the generalization is the addition and subtraction formula

$$\mu\big((a,b] \times (c,d] \,\big) \;=\; F(b,d) - F(a,d) - F(b,c) + F(a,c) \tag{5.4}$$

(*Hint*: Draw a picture in the plane.) For general k, the expansion of the product (5.3) is a sum with 2^k terms:

$$\mu\left(\prod_{j=1}^{k} (a_j, b_j] \right) \;=\; \sum_{\substack{c:\{\,1,\ldots,k\,\}\to R \\ c_j = a_j \text{ or } b_j \\ n_c = \#\{\, j: c_j = a_j \,\}}} (-1)^{n_c} \prod_{j=1}^{k} F_j(c_j) \tag{5.5}$$

If $F(x_1, x_2, \ldots, x_k)$ in (5.2) is not a product, the generalization of (5.5) is

Lemma 5.1 Let $C = \prod_{j=1}^{k} (a_j, b_j]$. Define $F(x_1, \ldots, x_k)$ by (5.2) where μ is a Borel measure on R^k with $\mu(R^k) < \infty$. Then

$$\Delta_C(F) \;=\; \mu\left(\prod_{j=1}^{k} (a_j, b_j] \right) \;=\; \sum_{\substack{c:\{\,1,\ldots,k\,\}\to R \\ c_j = a_j \text{ or } b_j \\ n_c = \#\{\, j: c_j = a_j \,\}}} (-1)^{n_c} F(c_1, c_2, \ldots, c_k) \tag{5.6}$$

Proof. By (5.4) and induction on k.

In particular, $F(x_1, x_2, \ldots, x_k)$ in (5.2) satisfies $\Delta_C(F) \geq 0$ for all cells $C \in \Gamma$ in (5.1). This is called the *box condition* for the function $F(x_1, x_2, \ldots, x_k)$.

The analog of right continuity for $F(x)$ for $x \in R^1$ is the following. We say that $x^n \downarrow x$ for $x^n = (x_1^n, x_2^n, \ldots, x_k^n)$ and $x = (x_1, x_2, \ldots, x_k) \in R^k$ if $x_j^n \downarrow x_j$ for each j, $1 \leq j \leq k$.

A function $F(x_1, x_2, \ldots, x_k)$ on R^k is *jointly right continuous* on R^k if $x^n \downarrow x \in R^k$ implies $F(x^n) \to F(x)$. If the box condition $\Delta_C(F) \geq 0$ holds for all $C \in \Gamma$, then $F(x^n) \downarrow F(x)$.

Exercises: **(1)** Show that $F(x_1, \ldots, x_n)$ defined by (5.2) is jointly right continuous.

(2) If $F(x_1, \ldots, x_k)$ in (5.2) satisfies the box condition $\Delta_C(F) \geq 0$ for all $C \in \Gamma$, then $x^n \downarrow x$ implies $F(x^n) \downarrow F(x)$.

Theorem 5.1 Let $F : R^k \to R$ be a function such that

(i) For all cells $C \in \Gamma$ in (5.1),

$$\Delta_C(F) \;=\; \sum_{\substack{c:\{\,1,\ldots,k\,\}\to R \\ c_j = a_j \text{ or } b_j \\ n_c = \#\{\,j: c_j = a_j\,\}}} (-1)^{n_c} F(c_1, c_2, \ldots, c_k) \;\geq\; 0 \qquad (5.7)$$

(ii) $F(x_1, x_2, \ldots, x_k)$ is jointly right continuous.

Then $\mu(C) = \Delta_C(F)$ defined by (5.7) is a premeasure on the semi-ring Γ in (5.1).

Note: Then, by the results in Section 3, $\mu(C)$ extends to a unique Borel measure $\mu(A)$ on R^k.

Proof of Theorem 5.1. It is not difficult to show (but with some work) that $\mu(C)$ is finitely additive on Γ by generalizing the proof that $\mu(C)$ defined by the product measure (5.3) is finitely additive on Γ. The results in Sections 2 and 3 carry over since they are about general semi-rings and outer measures. Now assume

$$C \;=\; \bigcup_{i=1}^{\infty} C_i, \qquad C,\, C_i \in \Gamma, \quad C_i \text{ disjoint}$$

Assume $C = \prod_{j=1}^{k}(a_j, b_j]$ and $C_i = \prod_{j=1}^{k}(a_{ij}, b_{ij}]$. Then

$$\mu(C) \;\geq\; \sum_{i=1}^{\infty} \mu(C_i)$$

follows from Lemma 2.2 as in the proof of Theorem 4.1. Define $C_\delta = \prod_{j=1}^{k}(a_j + \delta, b_j]$ and $C_{i\delta} = \prod_{j=1}^{k}(a_{ij}, b_{ij} + \delta]$. Then, by condition (ii) in Theorem 5.1, for all $\epsilon > 0$, there exist $\delta > 0$ and $\delta_i > 0$ such that

$$\mu(C_\delta) - \mu(C) < \epsilon, \qquad \mu(C_{i\delta_i}) - \mu(C_i) < \epsilon/2^i$$

for $1 \leq i < \infty$. This is the analog of (4.5) in the proof of Theorem 4.1. The rest of the proof of Theorem 4.1 carries over with changes only in the notation. Hence

$$\mu(C) \;=\; \sum_{i=1}^{\infty} \mu(C_i)$$

and $\mu(C)$ is a premeasure on Γ.

References.

1. Folland, G. B. (1999) *Real Analysis: Modern Techniques and Their Applications*, 2nd edn, John Wiley & Sons.