Hypergeometric Functions and Diffusion Processes

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1. Hypergeometric Functions. The hypergeometric function $F(a,b,c,x)$ is defined as the unique solution $y(x)$ of the equation

$$x(1-x)y'' + \big(c-(a+b+1)x\big)y' - aby = 0 \tag{1.1}$$

of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n, \qquad a_0 = 1 \tag{1.2}$$

(See, for example, Magnus, Oberhettinger, and Soni, 1966.) Substituting (1.2) into (1.1)

$$\begin{aligned}
&\sum_{n=1}^{\infty} n(n-1)a_n x^{n-1} - \sum_{n=0}^{\infty} n(n-1)a_n x^n \\
&\quad + \ c\sum_{n=0}^{\infty} n a_n x^{n-1} \ - \ (a+b+1)\sum_{n=0}^{\infty} n a_n x^n \ + \ ab\sum_{n=0}^{\infty} a_n x^n \\
&= \ \sum_{n=1}^{\infty} n(n-1+c)a_n x^{n-1} \ - \ \sum_{n=0}^{\infty}\big(n(n+a+b)+ab\big)a_n x^n \\
&= \ \sum_{n=0}^{\infty}\Big((n+1)(n+c)a_{n+1} - (n+a)(n+b)a_n\Big)x^n \ = \ 0
\end{aligned}$$

leads to the recurrence

$$a_{n+1} \ = \ \frac{(n+a)(n+b)}{(n+1)(n+c)}a_n$$

Assuming $c > 0$ and $a_0 = 1$,

$$\begin{aligned}
a_n \ &= \ \prod_{k=0}^{n-1}\frac{(k+a)(k+b)}{(k+c)(k+1)} \qquad (1.3)\\
&= \ \frac{a(a+1)\dots(a+n-1)b(b+1)\dots(b+n-1)}{c(c+1)\dots(c+n-1)\,n!} \\
&= \ \frac{a^{(n)}b^{(n)}}{c^{(n)}\,n!}, \qquad a^{(n)} = a(a+1)\dots(a+n-1)
\end{aligned}$$

Thus the hypergeometric function has the series representation

$$F(a,b,c,x) \;=\; \sum_{n=1}^{\infty} \frac{a^{(n)}b^{(n)}}{c^{(n)}n!} \;=\; 1 \;+\; \frac{ab}{c}x \;+\; \frac{a(a+1)b(b+1)}{c(c+1)}\frac{x^2}{2}$$
$$+\; \frac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)(c+2)}\frac{x^3}{6} \;+\; \ldots \tag{1.4}$$

In particular
- (i) $F(a,b,c,x) = F(b,a,c,x)$
- (ii) $F(a,b,c,x) = 1$ if and only if $a = 0$ or $b = 0$
- (iii) $F(a,b,c,x)$ is a nonconstant polynomial if and only if a or b is a negative integer
- (iv) In all other cases, $F(a,b,c,x)$ has radius of convergence exactly one.

By (1.4)

$$\begin{aligned} F(0,b,c,x) &= 1 \\ F(-1,b,c,x) &= 1 - \frac{b}{c}x \\ F(-2,b,c,x) &= 1 - \frac{2b}{c}x + \frac{2b(b+1)}{c(c+1)}\frac{x^2}{2} \end{aligned}$$

and in general

$$\begin{aligned} F(-n,b,c,x) &= \sum_{k=0}^{n} \frac{(-n)^{(k)}b^{(k)}}{c^{(k)}k!}x^k \\ &= \sum_{k=0}^{n} \frac{(-n)(-n+1)\ldots(-n+k-1)\,b^{(k)}}{c^{(k)}k!}x^k \\ &= \sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{b^{(k)}}{c^{(k)}}\frac{x^k}{k!} \end{aligned} \tag{1.5}$$

Special cases of polynomials (1.5) are

$$\begin{aligned} T_n(1-2x) &= F(-n,n,\frac{1}{2},x) \\ P_n(1-2x) &= F(-n,n+1,1,x) \\ C_n^{\lambda}(1-2x) &= \frac{(2\lambda)^{(n)}}{n!}F(-n,n+2\lambda,\,\lambda+\frac{1}{2},x) \\ P_n^{(\alpha,\beta)}(1-2x) &= \frac{(1+\alpha)^{(n)}}{n!}F(-n,n+\alpha+\beta+1,\,\alpha+1,x) \end{aligned}$$

where $T_n(x)$, $P_n(x)$, $C_n(x)$, and $P_n^{(\alpha,\beta)}(x)$ are the Chebyshev, Legendre, Gegenbauer, and Jacobi polynomials, respectively, defined on the interval $(-1,1)$. Note that every polynomial (1.5) can be written as a constant times a Jacobi polynomial for some values of α and β.

A large number of identities are known for $F(a,b,c,x)$, for example

$$F(a,b,c,x) \;=\; \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 y^{b-1}(1-y)^{c-b-1}(1-xy)^{-a}dy$$

which implies in particular

$$F(a,b,c,1) \;=\; \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

2. Jacobi Polynomials. The Jacobi polynomials are defined by

$$P_n^{(\alpha,\beta)}(x) \;=\; \frac{(1+\alpha)^{(n)}}{n!} F\Big(-n, n+\alpha+\beta+1,\ \alpha+1, \frac{1-x}{2}\Big) \tag{2.1}$$

so that $P_n^{(\alpha,\beta)}(1) = (1+\alpha)^{(n)}/n! = \Gamma(\alpha+n)/(\Gamma(\alpha+1)/n!)$. In particular

$$\begin{aligned} P_0^{(\alpha,\beta)}(x) &= 1 \\ P_1^{(\alpha,\beta)}(x) &= \left(\frac{\alpha-\beta}{2}\right) + \left(\frac{2+\alpha+\beta}{2}\right) x \end{aligned}$$

The Jacobi polynomials are orthogonal with respect to the weight $w(x) = (1-x)^\alpha(1+x)^\beta$ on $(-1,1)$ with

$$\begin{aligned} &\int_{-1}^1 P_n^{(\alpha,\beta)}(x) P_m^{(\alpha,\beta)}(x)\,(1-x)^\alpha(1+x)^\beta\,dx \\ &\qquad = \delta_{nm} \frac{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{n!\,\Gamma(\alpha+\beta+n+1)} \frac{2^{\alpha+\beta+1}}{\alpha+\beta+2n+1} \end{aligned}$$

or

$$\begin{aligned} &\int_0^1 P_n^{(\alpha,\beta)}(1-2x) P_m^{(\alpha,\beta)}(1-2x)\,x^\alpha(1-x)^\beta\,dx \\ &\qquad = \delta_{nm} \frac{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{(\alpha+\beta+2n+1)\,n!\,\Gamma(\alpha+\beta+n+1)} \end{aligned} \tag{2.2}$$

The Jacobi polynomials satisfy Rodrigues' formula

$$P_n^{(\alpha,\beta)}(x) \;=\; \frac{(-1)^n}{n!\,2^n} \frac{1}{(1-x)^\alpha(1+x)^\beta} \frac{d^n}{dx^n}\big((1-x)^{\alpha+n}(1+x)^{\beta+n}\big) \tag{2.3}$$

which implies

$$P_n^{(\alpha,\beta)}(x) \;=\; \frac{1}{2^n}\sum_{r=0}^{n}\binom{n+\alpha}{r}\binom{n+\beta}{n-r}(x+1)^r(x-1)^{n-r}$$

In particular

$$P_n^{(\alpha,\beta)}(-x) \;=\; (-1)^n P_n^{(\beta,\alpha)}(x)$$

and, if $\beta=\alpha$, $P_n^{(\alpha,\beta)}(x)$ is even if n is even and odd if n is odd. Also

$$P_n^{(\alpha,\beta)}(1-2x) \;=\; \sum_{r=0}^{n}(-1)^{n-r}\binom{n+\alpha}{r}\binom{n+\beta}{n-r}(1-x)^r x^{n-r}$$

By (2.1) and (1.1), the function $y(x)=P_n^{(\alpha,\beta)}(x)$ satisfies

$$(1-x^2)y''(x) + \big(\beta-\alpha-(\alpha+\beta+2)x\big)y'(x) + n(n+\alpha+\beta+1)y(x) \;=\; 0 \quad (2.4)$$

and $u(x)=(1-x)^\alpha(1+x)^\beta P_n^{(\alpha,\beta)}(x)$ satisfies

$$(1-x^2)u''(x) \;+\; \big(\alpha-\beta-(\alpha+\beta-2)x\big)u'(x) \;+\; (n+1)(n+\alpha+\beta)u(x) \;=\; 0$$

In particular, if $v(x)=u(1-2x)=Cx^\alpha(1-x)^\beta P_n^{(\alpha,\beta)}(1-2x)$,

$$x(1-x)v''(x) + \big(\beta-1-(\alpha+\beta-2)x\big)v'(x) + (n+1)(n+\alpha+\beta)v(x) \;=\; 0 \quad (2.5)$$

3. Gegenbauer Polynomials. Gegenbauer polynomials are defined by

$$\begin{aligned} C_n^\lambda(x) \;&=\; \frac{(2\lambda)^{(n)}}{n!}F\left(-n, n+2\lambda,\, \lambda+\frac{1}{2}, \frac{1-x}{2}\right) \\ &=\; \frac{(2\lambda)^{(n)}}{(\lambda+(1/2))^{(n)}}P^{(\lambda-1/2,\lambda-1/2)}(x) \end{aligned} \quad (3.1)$$

where $P_n^{(\alpha,\beta)}(x)$ are Jacobi polynomials. In particular

$$\begin{aligned} C_0^\lambda(x) \;&=\; 1 \\ C_1^\lambda(x) \;&=\; 2\lambda x \\ C_2^\lambda(x) \;&=\; 2\lambda(\lambda+1)x^2-\lambda \end{aligned} \quad (3.2)$$

The Gegenbauer polynomials are orthogonal with respect to the weight $w(x) = (1-x^2)^{\lambda-1/2}$ on $(-1,1)$ with

$$\int_{-1}^{1} C_n^\lambda(x) C_m^\lambda(x)\,(1-x^2)^{\lambda-1/2}\,dx \;=\; \delta_{nm}\frac{\pi 2^{1-2\lambda}\Gamma(n+2\lambda)}{(\lambda+n)\,n!\,\Gamma(\lambda)^2}$$

or

$$\int_0^1 C_n^\lambda(1-2x) C_m^\lambda(1-2x)\,\big(x(1-x)\big)^{\lambda-1/2}\,dx \\ =\; \delta_{nm}\frac{\pi 2^{1-4\lambda}\Gamma(n+2\lambda)}{(\lambda+n)\,n!\,\Gamma(\lambda)^2} \qquad (3.3)$$

The polynomials $C_n^\lambda(x)$ satisfy the recurrence relation

$$C_{n+2}^\lambda(x) \;=\; \frac{2(n+1+\lambda)}{n+2} x C_{n+1}^\lambda(x) \;-\; \frac{n+2\lambda}{n+2} C_n(x) \qquad (3.4)$$

By (2.4) and (2.5), the function $y(x) = C_n^\lambda(x)$ satisfies

$$(1-x^2)y''(x) \;-\; (2\lambda+1)xy'(x) \;+\; n(n+\lambda)y(x) \;=\; 0$$

and $u(x) = (1-x^2)^{\lambda-1/2} C_n^\lambda(x)$ satisfies

$$(1-x^2)u''(x) \;-\; (2\lambda-3)xu'(x) \;+\; (n+1)(n+2\lambda-1)u(x) \;=\; 0$$

In particular, if $v(x) = u(1-2x) = C\big(x(1-x)\big)^{\lambda-1/2} C_n^\lambda(1-2x)$,

$$x(1-x)v''(x) + (\lambda-3/2)(1-2x)v'(x) + (n+1)(n+2\lambda-1)v(x) \;=\; 0 \qquad (3.5)$$

4. Kimura's Expansion: Sturm-Liouville Theory for the Pure-Drift Equation. (This section and the next are essentially Kimura 1955 in a more modern and more mathematical framework.) The pure-drift equation in backwards form is

$$\frac{\partial}{\partial t}u(x,t) \;=\; (1/2)x(1-x)\frac{\partial^2}{\partial x^2}u(x,t) \;=\; L_x u(x,t) \qquad (4.1)$$

This is $L_x = (d/dm(x))(d/ds(x))$ in Feller or diffusion-process form where

$$s(x) = x \quad\text{ and }\quad m(dx) = \frac{2dx}{x(1-x)} \qquad (4.2)$$

are the scale function and speed measure of L_x. Since both boundaries of $(0,1)$ are pure-exit, the boundary conditions in (4.1) should be $u(0,t) = u(1,t) = 0$.

I claim that the inverse of the operator L_x in (4.1) on $I_0 = (0,1)$ with zero boundary conditions on $I = [0,1]$ is $-G_x$ where

$$G_x f(x) \;=\; \int_0^1 g(x,y) f(y) m(dy) \tag{4.3}$$

where in general

$$g(x,y) \;=\; \frac{s(x \wedge y)\,(s(1) - s(x \vee y))}{s(1) - s(0)}$$

where $x \wedge y = \min\{x,y\}$ and $x \vee y = \max\{x,y\}$, and since $s(x) = x$

$$\begin{aligned} g(x,y) \;&=\; x(1-y), \qquad 0 \le x \le y \le 1 \\ &=\; y(1-x), \qquad 0 \le y \le x \le 1 \\ &\le\; \min\bigl\{g(x,x),\, g(y,y)\bigr\} \end{aligned} \tag{4.4}$$

By (4.3) and (4.4)

$$G_x f(x) \;=\; (1-x)\int_0^x y f(y) m(dy) + x \int_x^1 (1-y) f(y) m(dy) \tag{4.5}$$

Given $f \in C(I)$, it follows from (4.5) that the unique solution of

$$L_x u(x) = (1/2)x(1-x)u''(x) = -f(x), \qquad u(0) = u(1) = 0 \tag{4.6}$$

for $u \in C^2(I_0) \cap C(I)$ is $u(x) = G_x f(x)$.

I claim that G_x is a Hilbert-Schmidt operator in the Hilbert space $L^2\bigl(I, dm\bigr)$. Clearly $g(x,y) = g(y,x)$, and by (4.4)

$$\begin{aligned} \int_0^1 g(x,y)^2 m(dy) \;&=\; (1-x)^2 \int_0^x y^2 \frac{2dy}{y(1-y)} \;+\; x^2 \int_x^1 (1-y)^2 \frac{2dy}{y(1-y)} \\ &\le\; (1-x)\int_0^x 2y\,dy + x\int_x^1 2(1-y)\,dy \\ &=\; (1-x)x^2 + x(1-x)^2 \;=\; x(1-x), \qquad 0 \le x \le 1 \end{aligned} \tag{4.7}$$

Thus

$$\int_0^1 \int_0^1 g(x,y)^2 m(dy) m(dx) \;<\; \int_0^1 x(1-x)\frac{2dx}{x(1-x)} \;=\; 2 \;<\; \infty \tag{4.8}$$

which is the condition that $g(x,y)$ be Hilbert-Schmidt.

It follows that there exists a complete orthogonal system of eigenvectors $v_n(x)$ in $L^2\big(I,dm\big)$ with eigenvalues μ_n satisfying the equations

$$G_x v_n(x) \;=\; \int_0^1 g(x,y)v_n(y)m(dy) \;=\; \mu_n v_n(x) \tag{4.9}$$

(Riesz and Nagy, 1955, Chapter 6). It follows from (4.5)–(4.6) that (4.9) is equivalent to

$$\mu_n(1/2)x(1-x)v_n''(x) = -v_n(x), \qquad v_n(0) = v_n(1) = 0 \tag{4.10}$$

for $v_n \in C^2(I_0)\cap C(I)$.

I claim that $\mu_n > 0$ in (4.9) and (4.10). If $v_n(x) \not\equiv 0$, we can assume that $\max_{0\le x\le 1} v_n(x) = v_n(x_1) > 0$ where $0 < x_1 < 1$. Then $v_n''(x_1) \le 0$, which implies that $\mu_n > 0$ in (4.10). Thus all eigenvalues $\mu_n > 0$ in (4.9) and (4.10), and (4.10) is equivalent to

$$(1/2)x(1-x)v_n''(x) = -\lambda_n v_n(x), \qquad v_n(0) = v_n(1) = 0 \tag{4.11}$$

for $\lambda_n = 1/\mu_n$.

Since $\mu_n > 0$ and $g(x,y) \in C(I\times I)$, it follows from Mercer's Theorem (Riesz and Nagy ibid.) that

$$g(x,y) \;=\; \sum_{n=1}^{\infty} \frac{v_n(x)v_n(y)}{\lambda_n Q_n}, \qquad Q_n = \int_0^1 v_n(z)^2 m(dz)$$

converges uniformly absolutely on I^2. This implies that

$$p(t,x,y) \;=\; \sum_{n=1}^{\infty} e^{-\lambda_n t}\frac{v_n(x)v_n(y)}{Q_n} \tag{4.12}$$

converges uniformly absolutely for $t \ge \epsilon > 0$ and $0 \le x,y \le 1$. It is proven in `OneDimDiffuseOps.tex` that

$$p(t,x,y) \ge 0, \qquad \int_0^1 p(t,x,z)m(dz) \le 1$$

for $t > 0$ and $0 \le x \le 1$, but this will not be needed in the following.

It follows from (4.7), (4.9), and Cauchy's inequality that

$$|v_n(x)| \;\le\; \lambda_n\sqrt{x(1-x)}\sqrt{Q_n} \tag{4.13}$$

Thus again by (4.9) and by (4.4) and (4.2)

$$\begin{aligned} |v_n(x)| &\le \lambda_n \int_0^1 g(x,y)|v_n(y)|m(dy) \\ &\le \lambda_n^2 \int_0^1 g(x,y)\sqrt{y(1-y)}m(dy)\sqrt{Q_n} \\ &= \lambda_n^2\, g(x,x) \int_0^1 \frac{2\sqrt{Q_n}}{\sqrt{y(1-y)}}dy \\ &= \lambda_n^2\, x(1-x)\pi\sqrt{Q_n} \end{aligned} \tag{4.14}$$

This implies by (4.12)

$$p(t,x,y) \le x(1-x)y(1-y)\,\pi\sum_{n=1}^{\infty}\lambda_n^4 e^{-\lambda_n t} \tag{4.15}$$

Since $\sum_{n=1}^{\infty}(1/\lambda_n^2) < 2$ by (4.8) and $\sum_{n=1}^{\infty}(1/\lambda_n) = 2$ by Mercer's Theorem and (4.4), this provides a second proof that the series (4.12) converges uniformly for $0 \le x, y \le 1$ and $t \ge \epsilon > 0$.

5. Eigenpolynomials of a Sturm-Liouville Expansion. By (3.5), the functions $v_n(x) = x(1-x)C_n(1-2x)$ for Gegenbauer polynomials $C_n(x) = C_n^{\lambda}(x)$ with $\lambda = 3/2$ satisfy

$$x(1-x)v_n''(x) + (n+1)(n+2)v_n(x) = 0 \tag{5.1}$$

with $v_n(0) = v_n(1) = 0$. Thus $v_n(x) = x(1-x)C_n(1-2x)$ are eigenfunctions of the equations (4.9) and (4.11) with eigenvalues $\lambda_n = (n+1)(n+2)/2$, where now $n \ge 0$ as opposed to $n \ge 1$ in (4.9), (4.11), and (4.12). In particular, the first few eigenvalues are $\lambda_0 = 1$, $\lambda_1 = 3$, $\lambda_2 = 6$, and $\lambda_3 = 10$.

Since the Gegenbauer polynomials $C_n^{3/2}(x)$ are a complete orthogonal system on $(-1,1)$ with respect to the measure $\nu(dx) = (1-x^2)dx$, the polynomials $v_n(x) = x(1-x)C_n(1-2x)$ are complete on I with respect to the speed measure $m(dx) = 2dx/(x(1-x))$ in (4.2).

By (3.3), the polynomials $v_n(x)$ satisfy the orthogonality relations

$$\begin{aligned} \int_0^1 v_n(x)v_m(x)\frac{2dx}{x(1-x)} &= \int_0^1 C_n(1-2x)C_m(1-2x)\,2x(1-x)\,dx \\ &= \delta_{mn}\frac{\pi 2^{2-6}(n+2)!}{(n+3/2)\,n!\,\Gamma(3/2)^2} \\ &= \delta_{mn}Q_n, \qquad Q_n = \frac{(n+1)(n+2)}{2(2n+3)} \end{aligned} \tag{5.2}$$

using the fact $\Gamma(3/2) = (1/2)\Gamma(1/2) = (1/2)\sqrt{\pi}$.

This implies that the transition density (4.12) with respect to speed measure $m(dx) = 2dx/(x(1-x))$ for the partial differential equation (4.1) is

$$p(t,x,y) \;=\; \sum_{n=0}^{\infty} e^{-(n+1)(n+2)t/2}\frac{2(2n+3)}{(n+1)(n+2)}v_n(x)v_n(y) \tag{5.3}$$

for $v_n(x) = x(1-x)C_n(1-2x)$.

We can use inequalities for special functions to obtain better inequalities for $v_n(x)$ than before. By Magnus *et al.* (1966),

$$\max_{-1\le y\le 1} |C_n^{\lambda}(y)| \;=\; \binom{n+2\lambda-1}{n} \;=\; \frac{(2\lambda)^{(n)}}{n!}$$

and thus

$$\max_{-1\le y\le 1} |C_n^{3/2}(y)| \;=\; \binom{n+2}{n} \;=\; \frac{(n+1)(n+2)}{2} \tag{5.4}$$

Thus since $v_n(x) = x(1-x)C_n(1-2x)$

$$|v_n(x)| \;\le\; x(1-x)\,\frac{(n+1)(n+2)}{2}$$

and hence

$$\frac{|v_n(x)|}{\sqrt{Q_n}} \;\le\; x(1-x)\sqrt{\frac{n+1)(n+2)(2n+3)}{2}} \tag{5.5}$$

This implies

$$\begin{aligned} p(t,x,y) \;&=\; \sum_{n=1}^{\infty} e^{-\lambda_n t}\frac{v_n(x)v_n(y)}{Q_n} \\ &\le\; x(1-x)y(1-y)\sum_{n=0}^{\infty} e^{-\lambda_n t}\lambda_n(2n+3) \end{aligned} \tag{5.6}$$

which provides a sharper inequality than (4.15).

In particular

$$\begin{aligned} \frac{p(t,x,y)}{x} \;&=\; \sum_{n=1}^{\infty} e^{-\lambda_n t}\frac{v_n(x)}{x}\frac{v_n(y)}{Q_n} \\ &=\; (1-x)\sum_{n=1}^{\infty} e^{-\lambda_n t}C_n(1-2x)\frac{v_n(y)}{Q_n} \end{aligned}$$

with uniform convergence for $0 \le x, y \le 1$ and $t \ge \epsilon > 0$. Thus

$$\begin{aligned}\frac{\partial p(t,0,y)}{\partial x} &= \sum_{n=1}^{\infty} e^{-\lambda_n t} C_n(1) \frac{v_n(y)}{Q_n} \\ &= \sum_{n=1}^{\infty} e^{-\lambda_n t} \lambda_n \frac{y(1-y)C_n(1-2y)}{Q_n} \end{aligned} \tag{5.7}$$

since $C_n^\lambda(1) = (2\lambda)^{(n)}/n!$ and $C_n(1) = C_n^{(3/2)}(1) = 3^{(n)}/n! = (n+2)!/(2n!) = (n+1)(n+2)/2 = \lambda_n$.

Since $C_0(x) = 1$ by (3.2), it follows from (5.3) that

$$p(t,x,y) \ \sim \ 3e^{-t} x(1-x)y(1-y)$$

as $t \to \infty$ uniformly for $0 \le x, y \le 1$. Similarly, the solution of

$$u_t = (1/2)x(1-x)u_{xx}, \qquad u(x,0) = u_0(x)$$

with $u(0,t) = u(1,t) = 0$ is

$$\begin{aligned} u(x,t) &= \int_0^1 p(t,x,y)u_0(y)m(dy) \\ &= \sum_{n=0}^{\infty} e^{-(n+1)(n+2)t/2} \frac{4(2n+3)}{(n+1)(n+2)} D_n v_n(x) \\ &\sim 6x(1-x)e^{-t} D_0 \qquad \text{as } t \to \infty \end{aligned} \tag{5.8}$$

where

$$D_n \ = \ (1/2)\int_0^1 u_0(y)v_n(y)m(dy) \ = \ \int_0^1 u_0(x)C_n(1-2x)dx \tag{5.9}$$

In particular, if ζ is the first-exit time from I, so that $P_x(\zeta > t) = \int_0^1 p(t,x,y)m(dy)$ in (5.8), then

$$u(x,t) \ = \ P(\zeta > t \mid X_0 = x) \ = \ 6x(1-x)e^{-t} + O\left(e^{-6t}\right) \tag{5.10}$$

The eigenvalue $\lambda_1 = 3$ does not enter (5.10) since the initial function $u_0(y)$ is even about $y = 1/2$ and hence $D_n = 0$ for odd n.

By (3.1), the polynomials $C_n(x) = C_n^{3/2}(x)$ satisfy

$$\begin{aligned} C_n(x) &= \frac{(n+1)(n+2)}{2} F\left(-n, n+3,\ 2,\ \frac{1-x}{2}\right) \\ &= \frac{(n+2)}{2} P^{(1,1)}(x) \end{aligned}$$

for Jacobi polynomials $P^{(1,1)}(x)$. For $\lambda = 3/2$ in (3.1), $3^{(n)} = (n+2)!/2$ and $2^{(n)} = (n+1)!$ so that $3^{(n)}/2^{(n)} = (n+2)/2$.

By (3.2) and (3.4), the $C_n(x)$ satisfy the recurrence

$$C_{n+2}(x) \;=\; \frac{2n+5}{n+2} x C_{n+1}(x) \;-\; \frac{n+3}{n+2} C_n(x) \tag{5.11}$$

with initial conditions $C_0(x) = 1$ and $C_1(x) = 3x$. In particular ($n = 0$), $C_2(x) = \frac{3}{2}(5x^2 - 1)$ so that $C_2(1-2x) = 6\big(1 - 5x(1-x)\big)$.

6. The Confluent Hypergeometric or Kummer's Function. The confluent hypergeometric function or Kummer's function $F(a, c, x)$ is defined as the unique solution $y(x)$ of the equation

$$xy'' + (c - x)y' - ay = 0 \tag{6.1}$$

of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n, \qquad a_0 = 1 \tag{6.2}$$

Substituting (6.2) into (6.1)

$$\begin{aligned}
&\sum_{n=1}^{\infty} n(n-1)a_n x^{n-1} \;+\; c\sum_{n=0}^{\infty} n a_n x^{n-1} \;-\; \sum_{n=0}^{\infty} n a_n x^n \;-\; a\sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n=1}^{\infty} n(n-1+c)a_n x^{n-1} \;-\; \sum_{n=0}^{\infty} (n+a)a_n x^n \\
&= \sum_{n=0}^{\infty} \Big((n+1)(n+c)a_{n+1} - (n+a)a_n\Big) x^n \;=\; 0
\end{aligned}$$

This leads to the recurrence

$$a_{n+1} \;=\; \frac{(n+a)}{(n+1)(n+c)} a_n$$

Assuming $c > 0$ and $a_0 = 1$,

$$\begin{aligned}
a_n \;&=\; \prod_{k=0}^{n-1} \frac{(k+a)}{(k+c)(k+1)} \;=\; \frac{a(a+1)\dots(a+n-1)}{c(c+1)\dots(c+n-1)\, n!} \qquad (6.3) \\
&= \frac{a^{(n)}}{c^{(n)}\, n!}, \qquad a^{(n)} = a(a+1)\dots(a+n-1)
\end{aligned}$$

Thus the confluent hypergeometric function has the series representation

$$F(a,c,x) \;=\; \sum_{n=1}^{\infty} \frac{a^{(n)}}{c^{(n)}n!} \;=\; 1 \;+\; \frac{a}{c}x \;+\; \frac{a(a+1)}{c(c+1)}\frac{x^2}{2}$$
$$+\; \frac{a(a+1)(a+2)}{c(c+1)(c+2)}\frac{x^3}{6} \;+\; \dots \tag{6.4}$$

In particular

(i) $F(a,c,x) = 1$ if and only if $a = 0$ or $b = 0$

(ii) $F(a,c,x)$ is a nonconstant polynomial if and only if a is a negative integer

(iii) $F(a,c,x)$ is always an entire function of x.

As in (1.4), $F(a,c,x)$ has the polynomial solutions

$$F(-n,b,c,x) \;=\; \sum_{k=0}^{n} \frac{(-n)^{(k)}}{c^{(k)}k!}x^k \;=\; \sum_{k=0}^{n} \frac{(-n)(-n+1)\dots(-n+k-1)}{c^{(k)}k!}x^k$$
$$= \sum_{k=0}^{n}(-1)^k \binom{n}{k} \frac{x^k}{c^{(k)}\,k!}$$

These do not appear to specialize to any classical polynomial systems. However, the moment generating function of a beta density

$$\begin{aligned}\varphi(s) \;&=\; \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^1 x^{a-1}(1-x)^{b-1}e^{sx}\,dx \\ &=\; \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{s^n}{n!}\frac{\Gamma(a+n)\Gamma(b)}{\Gamma(a+b+n)} \\ &=\; \sum_{n=0}^{\infty} \frac{s^n}{n!}\frac{a^{(n)}}{(a+b)^{(n)}} \;=\; F(a,a+b,s)\end{aligned}$$

can be expressed in terms of the confluent hypergeometric function.

7. Sturm-Liouville Expansions for the Selection-Drift Equation. The selection-drift equation is

$$\begin{aligned}\frac{\partial}{\partial t}u(x,t) \;&=\; (1/2)x(1-x)\frac{\partial^2}{\partial x^2}u(x,t) + \gamma x(1-x)\frac{\partial}{\partial x}u(x,t) \qquad (7.1)\\ &=\; L_x u(x,t)\end{aligned}$$

Here $L_x = (d/dm(x))(d/ds(x))$ for scale and speed measure

$$s(x) = \frac{1 - e^{-2\gamma x}}{2\gamma} \qquad \text{and} \qquad m(dx) = \frac{2e^{2\gamma x}dx}{x(1-x)} \tag{7.2}$$

where $s(0) = 0$ and $s'(0) = 1$. As in (4.3)–(4.8), the Green's function

$$g(x, y, \gamma) \;=\; \frac{s(x \wedge y)\,(s(1) - s(x \vee y))}{s(1) - s(0)}$$

satisfies

$$\int_0^1 \int_0^1 g(x, y, \gamma)^2 m(dy)m(dx) \;<\; \infty$$

This implies that, for fixed γ, the system

$$\int_0^1 g(x, y, \gamma)u(y)m(dy) \;=\; (1/\lambda)u(x)$$

or equivalently

$$(1/2)x(1-x)u''(x) + \gamma x(1-x)u'(x) = -\lambda u(x) \tag{7.3}$$

with $u(0) = u(1) = 0$ has a complete orthogonal set of eigenvectors $u_n(x)$ with respect to $m(dx)$ on I.

Consider solutions of (7.3) of the form

$$u(x, \gamma, \lambda) \;=\; \sum_{n=0}^{\infty} a_n(\gamma, \lambda)x^{n+r} \tag{7.4}$$

with $a_0(\gamma, \lambda) = 1$. Substituting (7.4) into (7.3)

$$\begin{aligned}
&\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-1} \;-\; \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} \\
&\quad +\; 2\gamma\sum_{n=0}^{\infty}(n+r)a_n x^{n+r} \;-\; 2\gamma\sum_{n=0}^{\infty}(n+r)a_n x^{n+r+1} \;+\; 2\lambda\sum_{n=0}^{\infty}a_n x^{n+r} \\
&= \;\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-1} \\
&\quad -\; \sum_{n=0}^{\infty}\big((n+r)(n+r-1-2\gamma) - 2\lambda\big)a_n x^{n+r} \;-\; 2\gamma\sum_{n=0}^{\infty}(n+r)a_n x^{n+r+1}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=-2}^{\infty}(n+r+2)(n+r+1)a_{n+2}x^{n+r+1} \\
&\quad - \sum_{n=-1}^{\infty}\big((n+r+1)(n+r-2\gamma)-2\lambda\big)a_{n+1}x^{n+r+1} \;-\; 2\gamma\sum_{n=0}^{\infty}(n+r)a_nx^{n+r+1} \\
&= r(r-1)a_0x^{r-1} + \Big((r+1)ra_1 - \big(r(r-1-2\gamma)-2\lambda\big)a_0\Big)x^r \\
&\quad + \sum_{n=0}^{\infty}\Big((n+r+2)(n+r+1)a_{n+2} \\
&\qquad - \big((n+r+1)(n+r-2\gamma)-2\lambda\big)a_{n+1} - 2\gamma(n+r)a_n\Big)x^{n+r+1} \;=\; 0
\end{aligned}
$$

Since $a_0 = 1$ by assumption, the first term of the indicial equation

$$r(r-1)a_0x^{r-1} + \Big((r+1)ra_1 - \big(r(r-1-2\gamma)-2\lambda\big)a_0\Big)x^r \;=\; 0$$

implies $r = 0$ or $r = 1$. However, $r = 0$ in the second term implies $\lambda = 0$, which violates $\lambda_n > 0$. The only solution with $\lambda \neq 0$ is $r = 1$. This has a unique solution with $a_0 = 1$, $a_1 = -(\gamma+\lambda)$, and

$$a_{n+2} \;=\; \frac{(n+2)(n+1-2\gamma)-2\lambda}{(n+3)(n+2)}a_{n+1} \;+\; \frac{2\gamma(n+1)}{(n+3)(n+2)}a_n \tag{7.5}$$

for $n \geq 0$.

By induction, $a_n = a_n(\gamma,\lambda)$ is a polynomial in γ and λ of degree n in either γ or λ unless $\lambda = ((m+2)(m+1-2\gamma)/2$ for some integer $m \geq 0$, in which case $a_n(\gamma,\lambda)$ is a polynomial of degree $n-1$ for $n \geq m-2$.

Note that $u(x,\gamma,\lambda)$ cannot be a polynomial in x for any $\gamma \neq 0$. This follows from the fact that $a_{n+2} = a_{n+1} = 0$ in (7.5) implies $a_n = 0$, and by induction $a_n = 0$ for all $n \geq 0$.

Writing (7.5) as $a_{n+2} = A_na_{n+1} + B_na_n$, we have

$$|A_n - 1| \leq \frac{D_M}{n}, \quad |B_n| \leq \frac{D_M}{n} \tag{7.6}$$

for constants $D_M > 0$ for $|\gamma| \leq M$, $|\lambda| \leq M$, and $n \geq 1$. Then by induction

$$
\begin{aligned}
|a_n(\gamma,\lambda)| \;&\leq\; C_0\prod_{k=1}^{n}\left(1+\frac{2D_M}{k}\right) \\
&=\; C_0\exp\left(\sum_{k=1}^{n}\log\left(1+\frac{2D_M}{k}\right)\right) \;\leq\; C_1n^{2D_M}
\end{aligned} \tag{7.7}
$$

uniformly for $|\gamma| \leq M$, $|\lambda| \leq M$, and $n \geq 1$. Thus the power series $u(x,\gamma,\lambda)$ in (7.4) has a radius of convergence at least one.

8. Eigenfunctions of the Selection-Drift Equation are Entire Functions. Given the solution $u(x)$ of $L_x u + \lambda u = 0$ in (7.4), we can find a second solution by setting $w(x) = u(x)v(x)$ and solving $L_x w + \lambda w = 0$ for $v(x)$. Specifically

$$\begin{aligned}(L_x + \lambda)w(x) \;&=\; v(x)(L_x + \lambda)u(x) \\ &\qquad +\; (1/2)x(1-x)\big(2v'u' + v''u + 2\gamma v'u'\big) \;=\; 0\end{aligned}$$

Since $L_x u + \lambda u = 0$, this is equivalent to $v''(x) + ((2u'(x)/u(x)) + 2\gamma)v'(x) = 0$. If $u(x) > 0$ for $0 < x < c$, this is solvable in that range with

$$w(x) \;=\; C_1 u(x) + C_2 u(x)\int_x^c \frac{e^{-2\gamma y}}{u(y)^2}\,dy \tag{8.1}$$

Recall $u(x) = x(1 + a_1 x + a_2 x^2 + \ldots)$ by (7.4), which corresponds to $C_2 = 0$ in (8.1). Thus a second linearly independent solution of $L_x u + \lambda u = 0$ is

$$\begin{aligned}u_2(x) \;&=\; u(x)\left(\frac{1}{x} + 2\lambda\log(1/x) + C_3 + \ldots\right) \\ &=\; \frac{u(x)}{x}\big(1 + 2\lambda x\log(1/x) + C_3 x + \ldots\big)\end{aligned} \tag{8.2}$$

Thus, in contrast with $u_1(0) = 0$ and $u_1'(0) = 1$ for the power-series solution (7.4) with $r = 1$, the second solution satisfies $u_2(0+) = 1$ and $u_2'(x) \sim 2\lambda\log(1/x)$ as $x \to 0$. We can use (8.1)–(8.2) to show

Theorem 8.1. Assume $\lambda = \lambda(\gamma) \neq 0$ and $u(x) = u(x, \lambda, \gamma)$ is a solution of the equation

$$(1/2)x(1-x)u''(x) + \gamma x(1-x)u'(x) + \lambda u(x) = 0 \tag{8.3}$$

for $0 < x < 1$. Then $u(0+) = u(1-) = 0$ if and only if $u(x)$ is an entire function of x, in which case $u(0) = u(1) = 0$.

Proof. If $u(x)$ is an entire function of x that is a solution of (8.3), then $u(0) = u(1) = 0$ by (8.3) since $\lambda \neq 0$. Conversely, let $u(x)$ be a solution of (8.3) for $0 < x < 1$ that satisfies

$$\lim_{\substack{x>0\\ x\to 0}} u(x) \;=\; \lim_{\substack{x<1\\ x\to 1}} u(x) \;=\; 0 \tag{8.4}$$

Note that (8.3) has two linearly independent solutions $u_1(x), u_2(x)$ in any open subinterval $(a, 1-a)$ for $a > 0$. It follows from (8.1) and (8.2) that any solution $u(x)$ of (8.3) that satisfies (8.4) can be analytically continued into a neighborhood of $x = 0$ with radius at least one, and by a similar argument $u(x)$ can be analytically continued into a neighborhood of $x = 1$ with radius at least one.

Write (8.3) in the form

$$u''(x) + 2\gamma u'(x) + \frac{2\lambda}{x(1-x)} u(x) = 0 \tag{8.4}$$

Since $x = 0$ and $x = 1$ are the only singularities of (8.3) in the form (8.4), it follows from the following lemma (Lemma 8.1) that $u(x)$ can be analytically continued to the entire complex plane. Thus $u(x)$ is entire, which completes the proof of Theorem 8.1 once we have proven Lemma 8.1.

Lemma 8.1. Consider the equation

$$y''(x) + a(x)y'(x) + b(x)y(x) = 0 \tag{8.5}$$

where $a(x)$ and $b(x)$ have power-series expansions about a point $x = x_0$ with radii of convergence at least $R > 0$. Then there are two linearly independent solutions of (8.5) for $|x - x_0| < R$ that are representable as power series about $x = x_0$ with radius of convergence at least R.

Proof of Lemma 8.1. Assume $x_0 = 0$ for definiteness. Write (8.5) as the system

$$\begin{pmatrix} y'(x) \\ y(x) \end{pmatrix}' = A(x) \begin{pmatrix} y'(x) \\ y(x) \end{pmatrix} \tag{8.6}$$

where $A(x)$ is the matrix-valued function

$$A(x) = \begin{pmatrix} -a(x) & -b(x) \\ 1 & 0 \end{pmatrix} = \sum_{n=0}^{\infty} A_n x^n$$

Here A_n are 2×2 matrices with $\|A_n\| \le C\rho^n$ for any $\rho < 1/R$, where $\|A_n\|$ means (for example) the sum of the absolute value of the matrix entries. By Picard iteration, the unique solution of (8.6) is

$$\begin{aligned} \begin{pmatrix} y'(x) \\ y(x) \end{pmatrix} &= \exp\left(\int_0^x A(y)dy \right) \begin{pmatrix} y'(0) \\ y(0) \end{pmatrix} \\ &= \exp\left(\sum_{n=0}^{\infty} A_n \frac{x^{n+1}}{n+1} \right) \begin{pmatrix} y'(0) \\ y(0) \end{pmatrix} \end{aligned}$$

where $\int_0^x A(y)dy$ means component by component and

$$e^B = I + B + \frac{B^2}{2} + \frac{B^3}{3!} + \ldots + \frac{B^n}{n!} + \ldots$$

This expresses the solution of (8.5) in terms of power series about $x = x_0$ with radius of convergence at least R, which completes the proof of Lemma 8.1.

Corollary 8.1. Suppose that $a(x) = a(x,z)$ and $b(x) = b(x,z)$ in Lemma 8.1 have convergent power series in x and z for $|x - x_0| < R$ and $|z| < M$. Then the solution $y(x) = y(x,z)$ also has a convergent power series in x and z for $|x - x_0| < R$ and $|z| < M$.

9. An Inequality for the Coefficients of $u(x,\gamma,\lambda)$. Recall that by definition in (7.4)

$$u(x,\gamma,\lambda) = x\sum_{n=0}^{\infty} a_n(\gamma,\lambda)x^n \tag{9.1}$$

where $a_n(\gamma,\lambda)$ are polynomials of degree n in γ and λ. Recall also that $|a_n(\gamma,\lambda)| \le Cn^{2D_M}$ for $|\gamma| \le M$ and $|\lambda| \le M$ by (7.7). The purpose here is to show that moreover

Theorem 9.1. If $a_n(\gamma,\lambda)$ are the coefficients in (9.1), then

$$|a_n(\gamma,\lambda)| \le \frac{C_M}{n^2} \tag{9.2}$$

uniformly for $|\gamma| \le M$ and $|\lambda| \le M$ for any $M < \infty$.

Remark. This implies that $u(x,\gamma,\lambda)$ is an entire function of γ and λ for any fixed x with $|x| \le 1$. In particular, for fixed γ, the eigenvalues λ_n are the zeros of an entire function $u(1,\gamma,\lambda)$.

Proof of Theorem 9.1. Let $a_{n+2} = A_n a_{n+1} + B_n a_n$ be the recurrence (7.5) and let $c_n = a_{n+1}/a_n$ if $a_n \ne 0$. Then

$$c_{n+1} = A_n + \frac{B_n}{c_n}, \qquad n \ge 1 \tag{9.3}$$

We break up the rest of the proof into several lemmas.

Lemma 9.1. For each (γ,λ), either

$$\text{(i)}\ \lim_{n\to\infty} c_n = 1 \qquad \text{or else} \tag{9.4a}$$

$$\text{(ii)}\ \lim_{n\to\infty} nc_n = -2\gamma \tag{9.4b}$$

If (9.4a) holds for some value $(\gamma,\lambda) = (\gamma_0,\lambda_0)$, then (9.4a) holds uniformly in some neighborhood of (γ_0,λ_0).

Proof of Lemma 9.1. Recall that $|A_n - 1| \le D/n$ and $|B_n| \le D/n$ by (7.6). For any $\delta > 0$, choose $n_\delta \ge (D + D\delta)/(\delta(1-\delta))$. Now assume $|c_n| \ge \delta$ for some $n \ge n_\delta$. Then by (9.3)

$$|c_{n+1}| \;\ge\; 1 - \frac{D}{n} - \frac{D}{n\delta} \;=\; 1 - \frac{1}{n\delta}(D + D\delta) \;\ge\; \delta$$

since this is equivalent to $1-\delta \ge (1/n\delta)(D+D\delta)$ or $n \ge (D+D\delta)/(\delta(1-\delta))$.

This implies that if $|c_n| \ge \delta$ for any $n \ge n_\delta$, then $|c_m| \ge \delta$ for all $m \ge n$. This in turn implies $\lim_{n\to\infty} c_n = 1$ since $A_n \to 1$, $B_n \to 0$, and $|1/c_n| \le 1/\delta$ in (9.3).

The only alternative is $\lim_{n\to\infty} |c_n| = 0$. Then $\lim_{n\to\infty} B_n/c_n = -1$ by (9.3) and $c_n \sim -2\gamma/n$ by (7.6) and (7.5). Thus, one of the alternatives (9.4) must occur.

To prove local uniformity, assume $\limsup_{n\to\infty} |c_n| > 0$. If then $|c_n| \ge 2\delta$ for some $n \ge (D + D\delta)/(\delta(1-\delta))$, then $\lim_{n\to\infty} c_n = 1$ as above. However, $|c_n(\gamma_0,\lambda_0)| \ge 2\delta$ implies $|c_n(\gamma,\lambda)| \ge \delta$ for $|\gamma - \gamma_0| < r$, $|\lambda - \lambda_0| < r$ for some $r > 0$. The argument above then shows that $\lim_{n\to\infty} c_n = 1$ uniformly for $|\gamma - \gamma_0| < r$ and $|\lambda - \lambda_0| < r$, which proves local uniformity.

Note that if (9.4b) holds, then $u(x,\gamma,\lambda)$ is automatically an entire function of x and hence λ is an eigenvalue by Theorem 8.1. Thus (9.4a) holds for each λ on the complement of a discrete countable set of λ. The next result is

Lemma 9.2. Assume that (9.4a) holds for some $(\gamma,\lambda) = (\gamma_0,\lambda_0)$. Then $|a_n(\gamma,\lambda)| \le C_M/n^2$ uniformly in some neighborhood of (γ_0,λ_0).

Proof of Lemma 9.2. By (7.5)

$$\begin{aligned} A_n &= \frac{(n+2)(n+1-2\gamma) - 2\lambda}{(n+3)(n+2)} \\ &= 1 + \frac{(n+2)(n+1-2\gamma) - 2\lambda - (n+3)(n+2)}{(n+3)(n+2)} \\ &= 1 + \frac{(3-2\gamma)n - 4\gamma - 2\lambda - 5n - 6}{(n+3)(n+2)} \\ &= 1 - \frac{2(1+\gamma)}{n} + O\left(\frac{1}{n^2}\right) \end{aligned} \tag{9.5}$$

and similarly

$$B_n \;=\; \frac{2\gamma(n+1)}{(n+3)(n+2)} a_n \;=\; \frac{2\gamma}{n} + O\left(\frac{1}{n^2}\right)$$

If $c_n \to 1$, then since $c_{n+1} = A_n + B_n/c_n$,

$$c_{n+1} \;=\; 1-\frac{2(1+\gamma)}{n}+\frac{2\gamma}{nc_n}+O\left(\frac{1}{n^2}\right)$$

This implies $c_n = 1 + O(1/n)$, which in turn implies

$$c_{n+1} \;=\; 1-\frac{2(1+\gamma)}{n}+\frac{2\gamma}{n}+O\left(\frac{1}{n^2}\right) \;=\; 1-\frac{2}{n}+O\left(\frac{1}{n^2}\right) \qquad (9.6)$$

By Lemma 9.1, (9.6) holds uniformly in a neighborhood of (γ_0, λ_0). In this neighborhood, for all $n \geq n_0$ and some constant $d \geq 0$,

$$1-\frac{2}{n}-\frac{d}{n^2} \;\leq\; \frac{a_{n+1}}{a_n} \;\leq\; 1-\frac{2}{n}+\frac{d}{n^2}$$

This implies

$$\prod_{k=n_0}^{n}\left(1-\frac{2}{k}-\frac{d}{k^2}\right) \;\leq\; \frac{a_{n+1}}{a_{n_0}} \;\leq\; \prod_{k=n_0}^{n}\left(1-\frac{2}{k}+\frac{d}{k^2}\right)$$

However

$$\begin{aligned}
\prod_{k=n_0}^{n}\left(1-\frac{2}{k}\pm\frac{d}{k^2}\right) \;&=\; \exp\left(\sum_{k=n_0}^{n}\log\left(1-\frac{2}{k}\pm\frac{d}{k^2}\right)\right) \\
&=\; \exp\left(-2\sum_{k=n_0}^{n}\frac{1}{k}+O\left(\sum_{k=n_0}^{n}\frac{1}{k^2}\right)\right) \\
&=\; \frac{1}{n^2}\exp\big(O(1)\big)
\end{aligned}$$

This completes the proof of Lemma 9.2.

Finally, since the set of λ such that (9.4b) holds for $\gamma = \gamma_0$ is discrete, we can choose a large $R > 0$ such that there are no eigenvalues λ for $\gamma = \gamma_0$ on $\partial N(0, R)$. Since $a_n(\gamma, \lambda)$ are polynomials in λ

$$a_n(\gamma,\lambda) \;=\; \frac{1}{2\pi i}\int_{\partial N(0,R)}\frac{a_n(\gamma,z)\,dz}{z-\lambda}, \qquad |\lambda| < R \qquad (9.7)$$

By assumption, the circle $|z| = R$ contains no eigenvalues for $\gamma = \gamma_0$. This implies by Lemma 9.2 and compactness that

$$|a_n(\gamma,z)| \;\leq\; \frac{C_R}{n^2}$$

uniformly for $|z| = R$ and $|\gamma - \gamma_0| < r$ for some $r > 0$. Then by (9.7)

$$|a_n(\gamma,\lambda)| \;\leq\; \frac{R}{2\pi}\frac{C_R}{n^2}\int_0^{2\pi}\frac{d\theta}{|Re^{i\theta}-\lambda|} \;\leq\; \frac{2C_R}{n^2}$$

uniformly for $|\lambda| < R/2$ and $|\gamma - \gamma_0| < r$. A second application of compactness completes the proof of Theorem 9.1.

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