One-Dimensional Diffusion Operators

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1. Basic Assumptions. Let $s(x)$ be a continuous, strictly-increasing function $s(x)$ on $I_0 = (0,1)$ and let $m(dx)$ be a Borel measure $m(dx)$ on I_0 (that is, $m(K) < \infty$ for compact subsets of $K \subset I_0$). Then we can define a "Feller-type operator"

$$Lf(x) = \frac{d}{dm(x)} \frac{d}{ds(x)} f(x) \tag{1.1}$$

for appropriate functions $f(x)$ on I_0. Here $s(x)$ is called the *scale function* of $Lf(x)$ and $m(dx)$ is the *speed measure*. (See e.g. Feller 1952, 1954, 1955, Ito and McKean 1965, Sawyer 1974.)

Assume further that

$$\begin{aligned} &s(x) \in C^2(I_0) \cap C(I), \quad s'(x) > 0, \quad s(dx) = s'(x)dx \\ &m(dx) = m(x)dx, \quad m(x) > 0, \quad m(x) \in C(I_0) \end{aligned} \tag{1.2}$$

for $0 < x < 1$ and $I = [0,1]$. Since $s(x) \in C(I)$, $s(0)$ and $s(1)$ are finite. (By $C(S)$ we mean the set of all continuous functions on S and by $C^k(S)$ for an open set S the set of all k-times-continuously differentiable functions on S.) Then

$$Lf(x) = \frac{1}{m(x)} \frac{d}{dx} \left(\frac{1}{s'(x)} f'(x) \right) \tag{1.3}$$

if $f(x) \in C^2(I_0)$. In the following we also assume that

$$\int_0^{1/2} \big(s(y) - s(0)\big) m(dy) < \infty, \qquad \int_{1/2}^1 \big(s(1) - s(y)\big) m(dy) < \infty \tag{1.4}$$

Condition (1.4) is called the condition that the operator $Lf(x)$ has "exit boundaries" at 0 and 1.

The motivating example that we have in mind is

$$s(x) = x \qquad \text{and} \qquad m(dx) = \frac{2dx}{x(1-x)}$$

and more generally

$$s(x) = \frac{1 - e^{-2\gamma x}}{2\gamma} \quad \text{and} \quad m(dx) = \frac{2e^{2\gamma x}dx}{x(1-x)} \tag{1.5}$$

for $\gamma \neq 0$. We leave it as an exercise to show that (1.3) and (1.5) correspond to the differential operator

$$Lf(x) \;=\; (1/2)x(1-x)f''(x) + \gamma x(1-x)f'(x)$$

2. An Integral Equation for $Lf(x)$. Define

$$\begin{aligned} g(x,y) \;&=\; \frac{\big(s(x\wedge y)-s(0)\big)\big(s(1)-s(x\vee y)\big)}{s(1)-s(0)} \qquad (2.1)\\ &=\; \begin{cases} \dfrac{\big(s(1)-s(x)\big)\big(s(y)-s(0)\big)}{s(1)-s(0)} & 0\le y\le x\le 1\\[2ex] \dfrac{\big(s(1)-s(y)\big)\big(s(x)-s(0)\big)}{s(1)-s(0)} & 0\le x\le y\le 1 \end{cases} \end{aligned}$$

where $x\wedge y=\min\{x,y\}$, and $x\vee y=\max\{x,y\}$. Then since $s(x)$ is increasing

$$\begin{aligned} g(x,y) \;&=\; g(y,x) \qquad (2.2)\\ g(x,x) \;&\le\; \min\{s(1)-s(x),\; s(x)-s(0)\} \;\le\; \frac{s(1)-s(0)}{4}\\ g(x,y) \;&\le\; \min\{g(x,x),\; g(y,y)\} \end{aligned}$$

Define the linear operator

$$\begin{aligned} Kf(x) \;&=\; \int_0^1 g(x,y)f(y)m(dy)\\ &=\; \frac{s(1)-s(x)}{s(1)-s(0)}\int_0^x \big(s(y)-s(0)\big)f(y)m(dy)\\ &\quad+\; \frac{s(x)-s(0)}{s(1)-s(0)}\int_x^1 \big(s(1)-s(y)\big)f(y)m(dy) \qquad (2.3) \end{aligned}$$

for $f(x)\in L^2(dm)=\{\,f(x):\int_0^1 f(x)^2m(dx)<\infty\,\}$. By Cauchy's inequality

$$\int_0^1 |g(y,y)h(y)|m(dy) \;\le\; \sqrt{\int_0^1 g(y,y)^2m(dy)}\,\sqrt{\int_0^1 h(y)^2m(dy)}$$

where $g(x,y)\le g(y,y)\le C$ and $\int_0^1 g(y,y)m(dy)<\infty$ by (1.4) and (2.2). Thus by dominated convergence each $Kf(x)\in C_0(I_0)$ where

$$C_0(I_0)=\{\,h(x)\in C(I_0):\lim_{x\to 0}h(x)=\lim_{x\to 1}h(x)=0\,\} \qquad (2.4)$$

The following theorem shows that the integral operator $Kf(x)$ in (2.3) is the inverse of the differential operator $Lf(x)$ in (1.1) and (1.3) with zero boundary conditions (that is, $Lf \in C_0(I_0)$).

Theorem 2.1. If $f(x) \in L^2(dm)$, then $h(x) = Kf(x)$ is the unique function $h(x) \in C^1(I_0) \cap C_0(I_0)$ such that $h'(x)$ is absolutely continuous in I_0 and $Lh(x) = -f(x)$ for $m(dx)$-almost every $x \in I_0$. If also $f \in C(I_0)$, then $h \in C^2(I_0)$ and $L_x h(x) = -f(x)$ for all $x \in I_0$.

Proof. Assume $h(x) = Kf(x)$ in (2.3). Thus

$$\begin{aligned} h(x) &= \frac{s(1)-s(x)}{s(1)-s(0)} \int_0^x \big(s(y)-s(0)\big) f(y) m(dy) \\ &\quad + \frac{s(x)-s(0)}{s(1)-s(0)} \int_x^1 \big(s(1)-s(y)\big) f(y) m(dy) \end{aligned}$$

As above, $h \in C(I)$ with $h(0) = h(1) = 0$. Also, $h(x)$ is absolutely continuous in x with

$$h'(x) = s'(x) \left(\int_x^1 \frac{s(1)-s(y)}{s(1)-s(0)} f(y) m(dy) \; - \; \int_0^x \frac{s(y)-s(0)}{s(1)-s(0)} f(y) m(dy) \right) \tag{2.5}$$

for almost every x. Since the right-hand side above is continuous, $h'(x)$ extends to a continuous function on $C(I_0)$ such that both $h'(x)$ and $h'(x)/s'(x)$ are themselves absolutely continuous. Moreover, $(d/dx)\big(h'(x)/s'(x)\big) = -f(x)m(x)$ for almost every x. Thus by (1.3) $Lh(x) = -f(x)$ for $m(dx)$-almost every $x \in I_0$. If $f \in C(I_0)$, then $h \in C^2(I_0)$ and $L_x h = -f$ by the same proof.

Conversely, suppose that $h_1(x) \in C^1(I_0) \cap C_0(I_0)$, $h_1'(x)$ is absolutely continuous, and $Lh_1(x) = -f(x)$ for almost every x. I now claim that $h_1 = h = Kf(x)$. First, let $g(x) = h(x) - h_1(x)$. Then, following the same convention as in (2.5), $g(x) \in C^1(I_0) \cap C_0(I_0)$, $g'(x)$ is absolutely continuous, and $L_x g(x) = Lh(x) - Lh_1(x) = 0$.

In generation, the relations (1.3) and $g \in C^1(I_0)$ with $g'(x)$ absolutely continuous implies

$$g'(x) = g'(x_0) + s'(x) \int_{x_0}^x L_x g(y) m(y) dy \tag{2.6}$$

for $0 < x, x_0 < 1$. Thus $g'(x) = 0$ if $L_x g(x) = 0$ for $m(dx)$-a.e. x and $g(x)$ is constant. Since $g(0) = g(1) = 0$, we must have $g = h - h_1 = 0$ and $h = h_1$, which completes the proof of Theorem 2.1.

We have actually proved a little more:

Lemma 2.1. Assume $g(x) \in C^1(I_0) \cap C(I)$, $g(0) = g(1) = 0$, $g'(x)$ is absolutely continuous, and

$$Lg(x) \;=\; \frac{1}{m(x)}\frac{d}{dx}\left(\frac{g'(x)}{s'(x)}\right) \;=\; -\lambda g(x) \tag{2.7}$$

for $m(dx)$-a.e. x and some constant λ. Then either $g(x) = 0$ for $0 < x < 1$ or else $\lambda > 0$.

Proof. By (1.3), (2.6) holds with $L_x g(y) = -\lambda g(y)$ for arbitrary $0 < x, x_0 < 1$. If $\lambda = 0$, then $g'(x) = 0$ for a.e. x and $g(x) = g(0) = g(1) = 0$. Now assume $\lambda \neq 0$.

Assume $\max_{0<x<1} g(x) = g(x_0) > 0$ and $\lambda < 0$. Then (2.6) implies that $g'(x) > 0$ for $x_0 < x < x_0 + \epsilon$ for some $\epsilon > 0$, which is inconsistent with the assumption that x_0 is an interior maximum of $g(x)$. Thus $\lambda > 0$.

Lemma 2.2. Assume $f(x) \in L^2(I, dm)$ and

$$Kf(x) = \int_0^1 g(x,y) f(y)\, m(dy) = \lambda f(x) \text{ a.e. } dm \tag{2.8}$$

for some constant λ. Then either $f(x) = 0$ a.e. dm. or else $\lambda > 0$.

Proof. By Theorem 2.1 and Lemma 2.1.

3. $\boldsymbol{Kf(x)}$ is a Hilbert-Schmidt Operator.

That is, I claim

Theorem 3.1. For $g(x,y)$ in (2.1)

$$\int_0^1 \int_0^1 g(x,y)^2\, m(dx)m(dy) \;<\; \infty \tag{3.1}$$

Proof. By (2.2) and (1.4)

$$k(x) = \int_0^1 g(x,y)^2\, m(dy) \;\le\; g(x,x) \int_0^1 g(y,y)\, m(dy) \;<\; \infty \tag{3.2}$$

Thus

$$\begin{aligned} \int_0^1 \int_0^1 g(x,y)^2\, m(dx)m(dy) \;&=\; \int_0^1 k(x) m(dx) \\ &\le \left(\int_0^1 g(y,y)\, m(dy)\right)^2 < \infty \end{aligned}$$

We can use a similar argument to obtain

Theorem 3.2. Suppose $f_n, f \in L^2(I, dm)$ and

$$\lim_{n\to\infty} \int_0^1 \big(f_n(y) - f(y)\big)^2 m(dy) \;=\; 0$$

and $Kf(x)$ is given by (2.3). Then

$$\lim_{n\to\infty} Kf_n(x) = K(x) \qquad \text{uniformly in } x$$

Proof. Then

$$\begin{aligned} |Kf_n(x) - Kf(x)|^2 \;&=\; \left| \int_0^1 g(x,y)\big(f_n(y) - f(y)\big)m(dy) \right|^2 \\ &\le\; \int_0^1 g(x,y)^2 m(dy) \int_0^1 \big(f_n(y) - f(y)\big)^2 m(dy) \\ &=\; k(x) \int_0^1 \big(f_n(y) - f(y)\big)^2 m(dy) \;\to\; 0 \end{aligned}$$

uniformly in x since $k(x)$ is bounded by (3.2)–(3.2).

4. Eigenfunctions and Eigenvalues of $\boldsymbol{Kf(x)}$. Since $Kf(x)$ is a (symmetric) Hilbert-Schmidt operator, or equivalently since $g(x,y)$ is a Hilbert-Schmidt kernel in $L^2(I, dm)$ by Theorem 3.1, there exist eigenfunctions $\phi_n(x)$ of K in $L^2(I, dm)$

$$K\phi_n(x) = \int_0^1 g(x,y)\phi_n(y)\, m(dy) = \mu_n \phi_n(x) \tag{4.1}$$

for nonzero real eigenvalues $\mu_n \neq 0$ satisfying the following conditions (see e.g. Riesz and Nagy, 1955, p242). The functions $\phi_n(x)$ are orthonormal, that is

$$(\phi_n, \phi_m) = \int_0^1 \phi_n(x)\phi_m(x)\, m(dx) = \delta_{mn} \tag{4.2}$$

where

$$(f, g) = \int_0^1 f(x)g(x)\, m(dx) \tag{4.3}$$

for $f, g \in L^2(I, dm)$, and any $f \in L^2(I, dm)$ can be written

$$f = g + \sum_{n=1}^{\infty} c_n \phi_n \tag{4.4}$$

where $Kg = 0$, $(g, \phi_n) = 0$ for all n, and the series converges in $L^2(I, dm)$ (Riesz and Nagy ibid.).

By Lemma 2.2 with $\lambda = 0$, $Kg = 0$ implies $g = 0$, so that $g = 0$ in (4.4). Thus the functions $\phi_n(x)$ are *complete orthonormal* in $L^2(I, dm)$ and $c_n = (f, \phi_n)$ in (4.4). That is, the expansion (4.4) with $g = 0$ holds for any $f \in L^2(I, dm)$ with $c_n = (f, \phi_n)$. Similarly, the eigenvalues $\mu_n > 0$ by Lemma 2.2.

By Theorem 2.1, the relations (4.1)–(4.2) implies $\phi_n(x) \in C^2(I_0) \cap C_0(I_0)$, $\phi_n(0) = \phi_n(1) = 0$, and $\phi_n(x) = -\mu_n L\phi_n(x)$. Thus (4.1) is equivalent to

$$L_x\phi_n(x) = -\lambda_n\phi_n(x), \quad \lambda_n = 1/\mu_n, \quad \phi_n(0) = \phi_n(1) = 0 \tag{4.5}$$

We can use the Hilbert-Schmidt property (3.1) and the properties of $\phi_n(x)$ to find out more information about the eigenvalues μ_n.

Theorem 4.1. The function $g(x, y)$ in (2.1) satisfies

$$g(x, y) \;=\; \sum_{n=1}^{\infty} \mu_n\phi_n(x)\phi_n(y) \tag{4.6}$$

with convergence in $L^2(I^2, dm^2)$. Moreover

$$\int_0^1\int_0^1 g(x, y)^2\, m(dx)m(dy) \;=\; \sum_{n=1}^{\infty} \mu_n^2 \;<\; \infty \tag{4.7}$$

Proof of Theorem 4.1. Since the functions $\phi_n(x)$ in (4.1) or (4.5) are complete orthonormal in $L^2(I_0, dm)$, the functions $\{\phi_m(x)\phi_n(y)\}$ are complete orthonormal in $L^2(I^2, dm^2)$ where $I^2 = I_0 \times I_0$ and $dm^2(dxdy) = m(dx)m(dy)$. This means that if $k(x, y)$ is any measureable function on the unit square with $\int_0^1\int_0^1 k(x, y)^2\, m(dx)m(dy) < \infty$, then

$$k(x, y) \;=\; \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c_{mn}\phi_m(x)\phi_n(y) \tag{4.8}$$

with convergence in $L^2(I^2, dm^2)$ and

$$c_{mn} \;=\; \int_0^1\int_0^1 g(w, z)\phi_m(z)\phi_n(w)m(dz)m(dw) \tag{4.9}$$

Moreover

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c_{mn}^2 \;=\; \int_0^1\int_0^1 k(x, y)^2\, m(dx)m(dy) \;<\; \infty \tag{4.10}$$

Since $g(x,y) = k(x,y)$ is in $L^2(I^2, dm^2)$ by (3.1), $g(x,y)$ satisfies (4.8) in the same sense. Since $\int g(x,y)\phi_n(y)m(dy) = \mu_n\phi_n(x)$ and $(\phi_n, \phi_m) = \delta_{mn}$, the Fourier coefficients in (4.9) are $c_{mn} = \mu_n\delta_{mn}$. This implies (4.6) and (4.7).

Theorem 4.2. Under the assumptions (1.1)–(1.4)

$$|\phi_n(x)| \;\le\; \lambda_n C_1 \sqrt{g(x,x)} \tag{4.11}$$

for an absolute constant C_1. If in addition $\int_0^1 \sqrt{g(x,x)}\, m(dx) < \infty$,

$$|\phi_n(x)| \;\le\; C_2 \lambda_n^2\, g(x,x) \tag{4.12}$$

Proof. By (4.1) and Cauchy's inequality

$$\begin{aligned} |\phi_n(x)| \;&\le\; \lambda_n \sqrt{\int_0^1 g(x,y)^2 m(dy)} \sqrt{\int_0^1 \phi_n(y)^2 m(dy)} \\ &\le\; \lambda_n \sqrt{g(x,x)} \sqrt{\int_0^1 g(y,y) m(dy)} \end{aligned} \tag{4.13}$$

since $g(x,y) \le \min\{g(x,x), g(y,y)\}$ by (2.2). The right-hand side of (4.13) is finite by (1.4). Similarly by (4.1) and (4.11)

$$\begin{aligned} |\phi_n(x)| \;&\le\; \lambda_n \int_0^1 g(x,y)|\phi_n(y)| m(dy) \\ &\le\; C_1 \lambda_n^2 g(x,x) \int_0^1 \sqrt{g(y,y)} m(dy) \end{aligned}$$

by (2.2).

We also have

Theorem 4.3. The series

$$g(x,y) \;=\; \sum_{n=1}^{\infty} \mu_n \phi_n(x) \phi_n(y) \tag{4.14}$$

converges uniformly absolutely for $0 \le x, y \le 1$. In particular by (4.14)

$$\int_0^1 g(x,x) m(dx) \;=\; \sum_{n=1}^{\infty} \mu_n \int_0^1 \phi_n(x)^2 m(dx) \;=\; \sum_{n=1}^{\infty} \mu_n \;<\; \infty \tag{4.15}$$

Proof. This is a special case of Mercer's Theorem (Riesz and Nagy, 1955, p245). We give the proof for completeness. First, consider the remainder

$$g_n(x,y) \;=\; g(x,y) \;-\; \sum_{k=1}^{n} \mu_k\phi_k(x)\phi_k(y) \;=\; \sum_{k=n+1}^{\infty} \mu_k\phi_k(x)\phi_k(y) \qquad (4.16)$$

with convergence in $L^2(I^2, dm^2)$. Since the $\phi_n(x) \in C_0(I)$, the function $g_n(x,y)$ is continuous in I^2 and satisfies

$$\int_0^1\int_0^1 f(x)f(y)g_n(x,y)m(dx)m(dy) \;=\; \sum_{k=n+1}^{\infty} \mu_k(f,\phi_k)^2 \;\geq\; 0 \qquad (4.17)$$

by (4.16). I claim that this implies $g_n(x,x) \geq 0$ for all $n \geq 0$ and x. For, if $g_n(x_0,x_0) < 0$ for some $x_0 \in I_0$, then $g_n(x,y) < 0$ for $x, y \in (x_0 - \epsilon, x_0 + \epsilon)$ for some $\epsilon > 0$. Let $f(x) \in L^2(I, dm)$ be any positive continuous function with support in the interval $(x_0 - \epsilon, x_0 + \epsilon)$. Then

$$\begin{aligned} &\int_0^1\int_0^1 f(x)f(y)g_n(x,y)m(dx)m(dy) \\ &\quad= \int_{x_0-\epsilon}^{x_0+\epsilon}\int_{x_0-\epsilon}^{x_0+\epsilon} f(x)f(y)g_n(x,y)m(dx)m(dy) \;<\; 0 \end{aligned}$$

which contradicts (4.17). It follows that $g_n(x,x) \geq 0$ and

$$0 \;\leq\; \sum_{k=1}^{n} \mu_k\phi_k(x)^2 \;\leq\; g(x,x) \qquad (4.18)$$

for $0 \leq x \leq 1$. Next, by Cauchy's inequality

$$\begin{aligned} \sum_{k=m}^{n} |\mu_k\phi_k(x)\phi_k(y)| \;&\leq\; \sqrt{\sum_{k=m}^{n} \mu_k\phi_k(x)^2}\,\sqrt{\sum_{k=m}^{n} \mu_k\phi_k(y)^2} \\ &\leq\; \left(\sqrt{\sum_{k=m}^{n} \mu_k\phi_k(x)^2}\right)\sqrt{g(y,y)} \qquad (4.19) \end{aligned}$$

by (4.18). This implies that, for each fixed x, the series

$$B(x,y) \;=\; \sum_{n=1}^{\infty} \mu_n\phi_n(x)\phi_n(y)$$

converges uniformly in y. Recall that $B(x,y) = g(x,y)$ a.e. dm^2 in the square by Theorem 4.1, but we will need pointwise identity.

For any $f(y) \in C_c(I_0)$ (that is, a continuous function of compact support $K \subset I_0$), the series

$$Bf(x) \;=\; \int_0^1 B(x,y)f(y)m(dy) \;=\; \sum_{n=1}^{\infty} \mu_n \phi_n(x)(f,\phi_n)$$

converges numerically for each x. By Theorem 3.2

$$Kf(x) \;=\; \int_0^1 g(x,y)f(y)m(dy \;=\; \sum_{n=1}^{\infty} \mu_n \phi_n(x)(f,\phi_n)$$

with uniform convergence in x. It follows that

$$\int_0^1 \big(B(x,y) - g(x,y)\big) f(y)m(dy) \;=\; 0$$

for every x and $f \in C_c(I_0)$. Since we can take $f(y) = (B(x,y) - g(x,y))\psi(y)$ where $\psi(y) \geq 0$ and $\psi \in C_c(I_0)$, we conclude $B(x,y) = g(x,y)$ for all (x,y). Thus

$$g(x,x) \;=\; B(x,x) \;=\; \sum_{n=1}^{\infty} \mu_n \phi_n(x)^2 \tag{4.20}$$

for all x with $0 < x < 1$, with (4.20) being trivial for $x = 0$ and $x = 1$. Since both $g(x,x)$ and $\phi_n(x)$ are continuous on the closed interval $[0,1]$, and since the convergence in (4.20) is monotonic since $\mu_n > 0$ and $\phi_n(x)^2 \geq 0$, it follows from Dini's theorem that the convergence in (4.20) is uniform in x. It then follows from (4.19) that the series (4.14) converges uniformly in I^2, which completes the proof of the theorem.

(A quick proof of Dini's Theorem: For each $x \in [0,1]$, the series converges within $2\epsilon > 0$ for $n \geq n_x$, and hence within $\epsilon > 0$ for $n \geq n_x$ and $|y - x| < \delta_x$ and $y \in I$ for some $\delta_x > 0$. Note $[0,1] \subset \cup_x (x - \delta_x, x + \delta_x)$ and use the Heine-Borel theorem.)

5. Diffusion Semi-Groups and Non-negativity. Define

$$p(t,x,y) \;=\; \sum_{n=1}^{\infty} e^{-\lambda_n t} \phi_n(x)\phi_n(y) \tag{5.1}$$

for $0 < t < \infty$ for the eigenvectors and eigenvalues $\phi_n(x)$ and λ_n defined in (4.5) and (4.1). Since $|\phi_n(x)| \leq C_1 \lambda_n$ by (4.11) where $\sum_{n=1}^{\infty}(1/\lambda_n) < \infty$, the series converges uniformly for $0 \le x, y \le 1$ and $a \le t < \infty$ for any $a > 0$. By Mercer's Theorem (Theorem 4.3), $\sum(1/\lambda_n)\phi_n(x)\phi_n(y)$ converges absolutely uniformly for $0 \le x, y \le 1$. Thus

$$\int_0^\infty p(t, x, y)dt = g(x, y) \tag{5.2}$$

with uniform convergence of the integral for $0 \le x, y \le 1$.

Since the series (5.1) converges in $L^2(I, dm)$ for each x,

$$p(t + s, x, z) \;=\; \int_0^1 p(t, x, y)p(s, y, z)m(dy) \tag{5.3}$$

for $t, s > 0$. Similarly, define

$$Q_t f(x) = \int_0^1 p(t, x, y)f(y)m(dy), \qquad f \in L^2(I, dm) \tag{5.4}$$

It follows from (5.3) that

$$Q_{t+s}f(x) = Q_t(Q_s f)(x), \qquad s, t > 0 \tag{5.5}$$

The main result of this section is

Theorem 5.1. For $p(t, x, y)$ in (5.1) and $t > 0$, $0 \le x \le 1$,

(i) $p(t, x, y) \ge 0 \quad (0 \le y \le 1)$

(ii) $\int_0^1 p(t, x, y)m(dy) \le 1$

(iii) For any $f \in C_0(I_0)$,

$$\lim_{t\to 0} \int_0^1 p(t, x, y)f(y)m(dy) = f(x) \tag{5.6}$$

uniformly for $0 \le x \le 1$.

Proof. Choose $h(x) \in C^2(I)$ such that, for some $a > 0$, $h(x) = 0$ for $0 \le x \le a$ and $1 - a \le x \le 1$ and set $f(x) = -L_x h(x)$. Then $f \in C(I)$ also satisfies $f(x) = 0$ for $0 \le x \le a$ and $1 - a \le x \le 1$.

Since $\mu_n = 1/\lambda_n$ in Theorem 4.3 (Mercer's theorem), the series

$$r(t, x, y) = \sum_{n=1}^\infty \frac{e^{-\lambda_n t}}{\lambda_n}\phi_n(x)\phi_n(y) \tag{5.7}$$

converges uniformly for $0 \le x, y \le 1$ and $0 \le t < \infty$. Also, $r(0,x,y) = \sum_{n=1}^{\infty}(1/\lambda_n)\phi_n(x)\phi_n(y) = g(x,y)$. Since $m(dy) = m(y)dy$ where $m(y)$ is bounded and continuous for $a \le y \le 1-a$, the function

$$\begin{aligned} u(x,t) &= \int_0^1 r(t,x,y)f(y)m(dy) \qquad (5.8) \\ &= \int_0^1 \left(\sum_{n=1}^{\infty} \frac{e^{-\lambda_n t}}{\lambda_n} \phi_n(x)\phi_n(y) \right) f(y)m(dy) \\ &= \sum_{n=1}^{\infty} \frac{e^{-\lambda_n t}}{\lambda_n} \phi_n(x) \int_0^1 \phi_n(y)f(y)m(dy) \end{aligned}$$

with uniform convergence for $0 \le x \le 1$ and $0 \le t < \infty$. Also

$$\begin{aligned} u(x,0) &= \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \phi_n(x) \int_0^1 \phi_n(y)f(y)m(dy) \\ &= \int_0^1 r(0,x,y)f(y)m(dy) = \int_0^1 g(x,y)f(y)m(dy) = h(x) \end{aligned}$$

by (4.14) and Theorem 2.1. Thus by (5.7), (4.15), and (4.5), the function $u(x,t)$ in (5.8) is continuous for $0 \le x \le 1$ and $t \ge 0$ and satisfies

$$\begin{aligned} &\frac{\partial}{\partial t}u(x,t) = L_x u(x,t), \qquad 0 < x < 1, \quad 0 < t < \infty \qquad (5.9) \\ &u(0,t) = u(1,t) = 0, \qquad 0 \le t < \infty \qquad (5.10) \\ &u(x,0) = h(x), \qquad 0 \le x \le 1 \end{aligned}$$

It follows from (4.1) and $g(x,y) = g(y,x)$ that

$$\begin{aligned} \int_0^1 \phi_n(x)h(x)m(dx) &= \int_0^1 \int_0^1 \phi_n(x)g(x,y)m(dx)\, f(y)m(dy) \\ &= \frac{1}{\lambda_n} \int_0^1 \phi_n(y)f(y)m(dy) \end{aligned}$$

Hence by (5.8)

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} e^{-\lambda_n t}\phi_n(x) \int_0^1 \phi_n(y)h(y)m(dy) \\ &= \int_0^1 \sum_{n=1}^{\infty} e^{-\lambda_n t}\phi_n(x)\phi_n(y)\, h(y)m(dy) \\ &= \int_0^1 p(t,x,y)h(y)m(dy) \qquad (5.11) \end{aligned}$$

By maximum principles for parabolic linear partial differential equations, any solution of (5.9) that is continuous for $0 \le x \le 1$ and $0 \le t \le T$ satisfies

$$\max_{0\le x\le 1,\ 0\le t\le T} u(x,t) \tag{5.12}$$
$$= \max\left\{ \max_{0\le t\le T} u(0,t),\ \ \max_{0\le t\le T} u(1,t),\ \ \max_{0\le x\le 1} u(x,0) \right\}$$

with a similar inequality for the minimum. (See Lemma 3, p167, of Protter and Weinberger, 1967.) If $0 \le h(x) \le 1$ in (5.10), this implies $0 \le u(x,t) \le 1$ for $0 \le x \le 1$ and $0 < t < \infty$.

We have now shown that $u(x,t)$ in (5.11) satisfies (5.12) for any $h \in C^2(I)$ with $h(x) = 0$ for $0 \le x \le a$ and $1-a \le x \le 1$ (some $a > 0$). If either $p(t_0,x_0,y_0) < 0$ or $\int_0^1 p(t_0,x_0,y)m(dy) > 1$ for any $t_0 > 0$ and $0 < x_0, y_0 < 1$, we quickly arrive at contradiction. This completes the proofs of parts (i) and (ii) of Theorem 5.1.

If $f(x)$ is smooth and vanishes near the endpoints, part (iii) of Theorem 5.1 follows from (5.11) and the arguments from (5.8) to (5.10). By part (ii)

$$|Q_t f(x)| \ \le\ \int_0^1 p(t,x,y)|f(y)|m(dy) \ \le\ \|f\| \ =\ \max_{0\le y\le 1} |f(y)| \tag{5.13}$$

for $f \in C[0,1]$. The uniformity in (5.13) implies that if (5.6) holds for all h is a subset $\mathcal{L} \subset C_0(I_0)$, then it also holds for all $h \in \overline{\mathcal{L}}$ with closure in the uniform norm on $(0,1)$. Thus (5.6) follows for all $f \in C_0(I_0)$. This completes the proof of Theorem 5.1.

6. Hille-Yoshida Theory. The linear operators Q_t defined in (5.4) satisfy $Q_{t+s} = Q_t Q_s$ by (5.5) and

$$\lim_{t\to 0} \|Q_t f - f\| \ =\ 0 \qquad \text{for all } f \in C_0(I_0) \tag{6.1}$$

by (5.6) where $\|f\| = \max_{0\le x\le 1} |f(x)|$. In other works, Q_t is a strongly-continuous semi-group of linear operators on the Banach space $B = C_0(I_0)$.

The infinitesimal generator of a strongly continuous semigroup of linear operators on a Banach space B is defined to be the linear operator on the subspace $\mathcal{D} = \mathcal{D}(A) \subseteq B$ defined by

$$\mathcal{D}(A) \ =\ \{\, h \in B : \lim_{t\to 0} \|(1/t)(Q_t h - h) - f\| \text{ for some } f \in B \,\} \tag{6.2}$$

with $Ah = f$. The operator A is linear on $\mathcal{D}(A)$, but generally $\mathcal{D}(A) \subset B$. In general $\overline{\mathcal{D}(A)} = B$ if Q_t is strongly continuous, since, for any $a > 0$ and $f \in B$, $h = \int_0^a Q_s f ds \in \mathcal{D}(A)$ with $Ah = Q_a f - f$.

If $\|Q_{t_0}f\| \le C\|f\|$ for all $f \in B$ and any $t_0 > 0$ and $C < \infty$, then exists constants $C_1 > 0$ and real K such that

$$\|Q_t f\| \;\le\; C_1 e^{Kt}\|f\| \tag{6.3}$$

for all $t < \infty$ and $f \in B$. If $\lambda > K$, the resolvent operator

$$R_\lambda f = \int_0^\infty e^{-\lambda t} Q_t f dt \tag{6.4}$$

is then a bounded linear operator on B.

Lemma 6.1. Let Q_t be a semigroup of linear operators on a Banach space B satisfying (6.1) and (6.3). Then, for any $\lambda > K$ for K in (6.3), R_λ maps B onto $\mathcal{D}(A)$ and

$$\lambda h - Ah = f \quad \text{whenever} \quad h = R_\lambda f$$

Proof. If $h = R_\lambda f$ for $f \in B$, then

$$\begin{aligned} Q_t h \;&=\; \int_0^\infty e^{-\lambda s} Q_{t+s} f ds \;=\; e^{\lambda t}\int_t^\infty e^{-\lambda s} Q_s f ds \\ &=\; e^{\lambda t} h - e^{\lambda t}\int_0^t e^{-\lambda s} Q_s f ds \end{aligned}$$

Thus $(1/h)(Q_t h - h) \to \lambda h - f$ as $t \to 0$. This implies that, for any $f \in B$, $h = R_\lambda f \in \mathcal{D}(A)$ and $Ah = \lambda h - f$, or $(\lambda - A)h = f$. In particular, (i) the range of R_λ is contained in the domain $\mathcal{D}(A)$, (ii) the range of $\lambda\, I - A$ is all of B, and (iii) $(\lambda - A)R_\lambda f = f$ for all $f \in B$.

Conversely, assume $h_1 \in \mathcal{D}(A)$. Set $f = (\lambda - A)h_1$ and $h = R_\lambda f$. Then $(\lambda - A)h = f = (\lambda - A)h_1$ and, if $g = h - h_1$, $g \in \mathcal{D}(A)$ and $(\lambda - A)g = 0$. In general if $g \in \mathcal{D}(A)$ and $s > 0$, then

$$(1/t)(Q_t - I)Q_s g = (1/t)(Q_{t+s} - Q_s)g = Q_s\big((1/t)Q_t g - g\big) \to Q_s A g$$

as $t \to 0$. This implies that $Q_s g \in \mathcal{D}(A)$ and $AQ_s = Q_s A$. If $Ag = \lambda g$, then $Q_t g \in \mathcal{D}(A)$ for all $t > 0$ and $A(Q_t g) = Q_t A g = \lambda Q_t g$. This implies $Q_t g = e^{\lambda t} g$, which violates (6.3) if $g \ne 0$ and $\lambda > K$. Thus $g = 0$, which implies $h_1 = h$. Hence the range of R_λ is all of $\mathcal{D}(A)$, which completes the proof of the lemma.

We can find A and $\mathcal{D}(A)$ exactly for the semigroup Q_t defined by (5.4) in Section 5 for $B = C_0(I_0)$.

Theorem 6.1. Let $L_x = \big(d/dm(x)\big)\big(d/ds(x)\big)$ be the operator in (1.1) and let Q_t be as in (5.4) for $B = C_0(I_0)$. Then

$$\mathcal{D}(A) \;=\; \{\, h : h \in C^2(I_0) \cap C_0(I_0) \;\text{ and }\; L_x h \in C_0(I_0)\,\} \tag{6.6}$$

with

$$Ah(x) = L_x h(x) = \frac{d}{dm(x)}\frac{d}{ds(x)}h(x)$$

Proof. It follows from (5.1), (4.11), and Theorem 5.1 that

$$|Q_t f(x)| \;=\; \left|\int_0^1 p(t,x,y)f(y)m(dy)\right| \;\le\; C\sum_{n=1}^{\infty}\lambda_n^2 e^{-\lambda_n t}\,\|f\|$$

for $f \in C(I)$ and $0 \le t < \infty$. Since $\psi(t) = \big(\sum_{n=1}^{\infty}\lambda_n^2 e^{-\lambda_n t}\big)/e^{-\lambda_1 t}$ is bounded and continuous for $1 \le t < \infty$ with $\lim_{t\to\infty}\psi(t) = C$ where C is positive and finite, it follows that

$$|Q_t f(x)| \;\le\; Ce^{-\lambda_1 t}\|f\|$$

for $0 \le t < \infty$. Thus (6.3) holds with $K = -\lambda_1$. Hence by Lemma 6.1 with $\lambda = 0$

$$R_0 f(x) = \int_0^{\infty} Q_s f(s)ds = \int_0^1 g(x,y)f(y)m(dy)$$

by (5.2). Hence $\mathcal{D}(A)$ is the range of R_0 on $B = C_0(I_0)$, which by Theorem 2.1 is exactly $\mathcal{D}(A)$ in (6.6). By Lemma 6.1, $h \in \mathcal{D}(A)$ if and only if $h = R_0 f$ for some $f \in B$ with $Ah = -f$, so that $Ah = L_x h$ by Theorem 2.1.

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