Poisson Random Fields — Exercises

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1. Let $N = (N_1, N_2, \dots, N_n)$ be n independent Poisson random variables with $E(N_i) = c_i$. (Recall that N is Poisson with mean c if N is an integer-valued random variable with $P(N = n) = e^{-c}c^n/n!$ for $n = 0, 1, \dots$.) Prove that

$$E\left(e^{\Sigma_{i=1}^n u_i N_i}\right) \;=\; e^{\Sigma_{i=1}^n c_i\left(e^{u_i}-1\right)} \tag{1}$$

for all choices of numbers u_i. Conversely, if $N = (N_1, N_2, \dots, N_n)$ is an arbitrary set of n random variables such that (1) holds for all u_i, prove that the N_i are independent Poisson random variables with means $E(N_i) = c_i$. (*Hint*: Use the fact that if $E(e^{\Sigma_{i=1}^r u_i X_i}) = E(e^{\Sigma_{i=1}^r u_i Y_i}) < \infty$ for $-a < u_i < a$ for some $a > 0$ for two random vectors $X = (X_1, \dots, X_r)$ and $Y = (Y_1, \dots, Y_r)$, then X and Y have the same probability distribution.)

2. Let $N = (N_1, \dots, N_n)$ be as in problem 1. Define a measure $\mu(A)$ on the set $X = \{\,1, 2, \dots, n\,\}$ by $\mu(\{i\}) = c_i$, so that $\mu(A) = \sum_{i\in A} c_i$. Use the vector N to define a random measure $N(A)$ on X by $N(\{i\}) = N_i$, so that $N(A) = \sum_{i\in A} N_i$ for $A \subseteq X$. Thus the random measure $N(A)$ is purely atomic with the (random) integer-valued atom N_i at $i \in X$. Prove that

$$E\left(e^{\int_X f(y)N(dy)}\right) \;=\; e^{\int_X \left(e^{f(y)}-1\right)\mu(dy)} \tag{2}$$

for all bounded functions $f(y)$ on X. Conversely, prove that if (2) holds for all bounded functions $f(y)$ on X, then N_i are independent Poisson random variables with means $E(N_i) = c_i$.

Definition. A *Poisson random field* is a random measure $\big(X, \mathcal{F}, N(A)\big)$ on a measure space $(X, \mathcal{F})$ (that is, $N(A)$ are random variables such that, with probability one, $\big(X, \mathcal{F}, N(A)\big)$ is a measure) with *mean measure* $(X, \mathcal{F}, \mu)$ if (2) holds whenever $f(y)$ is a bounded $\mathcal{F}$-measureable function on X with $\int_X |f(y)|\,\mu(dy) < \infty$.

3. Assuming that we have been able to construct a Poisson random field $\big(X, \mathcal{F}, N(A)\big)$ for a mean measure $(X, \mathcal{F}, \mu)$, use (2) to prove

(i) If $\mu(A) < \infty$ for some $A \in \mathcal{F}$, then $N(A)$ is Poisson with mean $\mu(A)$. (*Hint*: Try $f(y) = uI_A(y)$ in (2).)

(ii) If $\mu(A) < \infty$ and $\mu(B) < \infty$ for two disjoint sets $A, B \in \mathcal{F}$, then $N(A)$ and $N(B)$ are independent. (Note that $N(A \cup B) = N(A) + N(B)$ since N is a random measure by definition.)

(iii) If $\{ A_n \}$ are disjoint with $A_n \in \mathcal{F}$ and $\mu(A_n) < \infty$, then $\{ N(A_n) \}$ are independent. (Recall that an infinite set of random variables is independent if and only if every finite subset is independent.)

(iv) If $\mu(A) = \infty$ for some $A \in \mathcal{F}$ and A is sigma-finite (that is, $A = \cup_{i=1}^{\infty} A_n$ where $A_n \in \mathcal{F}$, $\{A_n\}$ are disjoint, and $\mu(A_n) < \infty$), then $N(A) = \infty$ almost surely. (*Hint*: By assumption, $N(A) = \sum_{i=1}^{\infty} N(A_n)$ almost surely. Consider the random events $C_n = \{ N(A_n) \geq 1 \}$ and see if you can use the Borel-Cantelli lemma. Are the $\{ C_n \}$ independent?)

4. Let $N = (N_1, N_2, \ldots, N_n)$ be as in Problem 1. Suppose that there are N_i objects of some kind at i $(1 \leq i \leq n)$. At a particular time, all of the objects move independently of one another to points in a finite set Y. Assume that objects at i $(1 \leq i \leq n)$ move to $y \in Y$ with probability $\pi(i, y)$, where $\pi(i, y) \geq 0$ and $\sum_{y \in Y} \pi(i, y) = 1$ for each i.

Let M_y be the total number of objects that end up at $y \in Y$ (that is, from all starting points i). Prove that $\{ M_y : y \in Y \}$ are independent Poisson random variables and find $E(M_y)$. Does the form of $E(M_y)$ seem plausible? (*Hint*: Verify (1) by calculating

$$E\left(e^{\Sigma_{y \in Y} v_y M_y}\right) \;=\; E\left(E\left(e^{\Sigma_{y \in Y} v_y M_y} \mid N\right)\right) \tag{3}$$

for $N = (N_1, N_2, \ldots, N_n)$. Note that, given N, the positions of the objects in Y are independent Y-valued random variables with probability distributions depending on their initial position.)

5. Let $N = (N_1, N_2, \ldots, N_n)$ be as in Problem 1 and suppose that there are N_i objects at i, as in Problem 4. Suppose that, independently for all objects, each object at i is colored green with probability g_i and colored red with probability r_i, where $g_i + r_i \leq 1$. If an object is not colored either red or green, then it remains its original dull gray color. Let G_i be the number of green objects and R_i be the number of red objects at i.

(i) Prove that the $\{ G_1, G_2, \ldots, G_n \}$ are independent Poisson random variables. Find $E(G_i)$. (This is called a *thinning* of $\{ N_1, N_2, \ldots, N_n \}$.)

(ii) Prove that $\{ G_1, \ldots, G_n, R_1, \ldots, R_n \}$ together are independent Poisson random variables. (In this case, $\{ G_1, G_2, \ldots, G_n \}$ and $\{ R_1, R_2, \ldots, R_n \}$ are called *orthogonal thinnings* of $\{ N_1, N_2, \ldots, N_n \}$.)

(iii) Let $G = \sum_{i=1}^{n} G_i$ and $R = \sum_{i=1}^{n} R_i$. Prove that G and R are independent Poisson random variables. Find $E(G)$ and $E(R)$.

(*Hint*: Apply Problem 4 for an appropriate choice of a set Y. Apply (2) on Y for various functions $f(y)$. What set Y did you choose?)

6. Let $\pi(i,j)$ be a Markov transition function on the set $X = \{0,1,2,\ldots,K\}$ for some integer K. Assume that $\{1,2,\ldots,K-1\}$ are transient states and that $0, K$ are traps.

Suppose that, at each time $n = 0,1,2,\ldots$, a random number V_n of objects is placed at state $1 \in X$, where $\{V_n : n \geq 0\}$ are independent Poisson random variables with $E(V_n) = v > 0$ and there were no objects in X before time 0. Immediately after the n^{th} set of immigrants arrive ($n \geq 0$), all objects move one step independently according to $\pi(i,j)$. In particular, the probability that one of the objects that originally arrived at 1 at time n is at i at time $n+t$ is given by $\pi^{(t)}(1,i)$, where $\pi^{(t)}$ is the t^{th} matrix power of $\pi(i,j)$.

Let $N(n,i)$ be the number of objects at i at time $n \geq 0$, where the population is counted in the n^{th} time step just after the arrival of the new immigrants. (Thus $N(0,1) = V_0$ and $N(n,1) \geq V_n \geq 0$.) Then

(i) Prove that, for each n, $\{N(n,i)\}$ are independent Poisson for $0 \leq i \leq K$.

(ii) Prove that $\{N(n,i) : 1 \leq i \leq K-1\}$ converges in distribution as $n \to \infty$ to independent Poisson random variables $N(i)$ $(1 \leq i \leq K-1)$. Find $E\big(N(i)\big)$.

(iii) Find an expression for

$$\nu_K = \lim_{n\to\infty} \frac{1}{n} E\big(N(n,K)\big)$$

and explain what ν_K means in terms of the objects. (*Hint*: It is the asymptotic rate at which objects are trapped at K.)

(*Hints*: Note that

$$N(n,i) = \sum_{t=0}^{n} N(t,n,i) \tag{4}$$

where $N(t,n,i)$ are the objects that began at state 1 at time $t \leq n$ and then ended up at i at time n. (In particular, $\sum_{i=0}^{K} N(t,n,i) = V_t$ for $n \geq t$, since the objects don't leave $X = \{0,1,2,\ldots,K\}$.) Argue as in Problem 4 to show that, for fixed n and t, $\{N(t,n,i)\}$ are independent Poisson for $0 \leq i \leq K$. Use the independence of the V_t to conclude that $\{N(t,n,i)\}$ are also independent for $0 \leq t \leq n$ and $0 \leq i \leq K$. Use (4) to conclude that $\{N(n,i)\}$ $(0 \leq i \leq K)$ are independent Poisson for each n. Finally, use either (1) or (2) and take limits as $n \to \infty$.)