# Sample Path Regularity for One-Dimensional Diffusion Processes

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**1. Basic Assumptions.** Assume that

(i) $\{T_t\}$ is a strongly-continuous semigroup of bounded linear operators on the Banach space

$$B = C_0(I) = \{\, f \text{ continuous on } I = [0,1] : f(0) = f(1) = 0 \,\}$$

(ii) If $f(x) \ge 0$ and $f \in B$, then $T_t f(x) \ge 0$ and $T_t f(x) \le \max_{y\in I} f(y)$.

In general, the infinitesimal generator of a strongly-continuuous semigroup of linear operators $T_t$ on any Banach space $B$ is defined by

$$Af = \lim_{h\to 0} (T_h f - f)/h \tag{1.1}$$

on the set

$$\mathcal{D} = \mathcal{D}(A) = \{f : \text{The limit in (1.1) exists in the norm of } B\}$$

Suppose further that

(iii) The operator $A$ and set $\mathcal{D}(A)$ satisfy the following conditions. Let

$$Lf(x) = \frac{d}{dm(x)}\frac{d}{ds(x)} f(x) \tag{1.2}$$

for $f \in C^2(I_0)$, where $C^2(I_0)$ is the set of all functions on $I_0 = (0,1)$ that are twice-continuously differentiable on $I_0$. In (1.2), $s(x)$ is strictly increasing, continuously differentiable, and bounded on $(0,1)$, $s'(x) > 0$ on $(0,1)$, and the measure $m(dx) = m(x)dx$ where $m(x)$ is continuous and $m(x) > 0$ on $(0,1)$. Thus we can also write

$$Lf(x) = \frac{1}{m(x)}\frac{d}{dx}\,\frac{1}{s'(x)}\frac{d}{dx}\, f(x) \tag{1.3}$$

If $f \in C^2(I_0) \cap C_0(I)$ and $Lf \in C_0(I)$, then $f \in \mathcal{D}(A)$ and $Af = Lf$.

(iv) Since $s(x)$ is strictly increasing and bounded on $(0,1)$, we can assume without loss of generality that $s(0) = s(0+) = 0$ and $s(1) = s(1-) = 1$. We also assume

$$\int_0^{x_0} s(x)m(dx) \;+\; \int_{x_0}^1 \big(1 - s(x)\big)m(dx) \;<\; \infty \tag{1.4}$$

whenever $0 < x_0 < 1$. In terms of diffusion process theory, this is the condition that the endpoints are regular or exit.

**2. Some Useful Identities.** **(1)** Define

$$g(x,y) = \frac{\big(s(1)-s(x\vee y)\big)\big(s(x\wedge y)-s(0)\big)}{s(1)-s(0)} \tag{2.1}$$

where $0\le x,y\le 1$, $x\wedge y=\min\{x,y\}$, and $x\vee y=\max\{x,y\}$. Let

$$\begin{aligned} f(x) &= \int_0^1 g(x,y)h(y)m(dy) && (2.2)\\ &= \frac{s(1)-s(x)}{s(1)-s(0)}\int_0^x (s(y)-s(0))h(y)m(dy) \\ &\quad + \frac{s(x)-s(0)}{s(1)-s(0)}\int_x^1 (s(1)-s(y))h(y)m(dy) && (2.3) \end{aligned}$$

where $h(y)$ is bounded on $I_0$. By (2.1)

$$\begin{aligned} g(x,y) &\le \min\{g(x,x),\, g(y,y)\} && (2.4)\\ g(y,y) &\le \min\{s(1)-s(y),\, s(y)-s(0)\} \end{aligned}$$

If $h(y)\ge 0$, then by (2.2) and (2.4)

$$\begin{aligned} f(x) &\le \int_0^1 g(y,y)h(y)m(dy) && (2.5)\\ &\le \int_0^1 \min\{s(1)-s(y),\, s(y)-s(0)\}h(y)m(dy) \\ &= \int_0^{x_0} \big(s(y)-s(0)\big)h(y)m(dy) \;+\; \int_{x_0}^1 \big(s(1)-s(y)\big)h(y)m(dy) && (2.6) \end{aligned}$$

where $x_0$ is determined by $s(x_0)-s(0)=(1/2)\big(s(1)-s(0)\big)$. It then follows from (1.4) that $f(x)$ in (2.2) is uniformly bounded with

$$|f(x)| \;\le\; C\,\max_y |h(y)| \tag{2.7}$$

where $C$ is the constant on the right-hand-side value of (2.6) if $h(y)$ is replaced by 1. It follows similarly from (2.4) and the dominated convergence theorem that $f(x)\in C(I)$ with $f(0)=f(1)=0$.

**(2)** If $h(x)\in C_0(I)$, then $f(x)$ in (2.2) is continuous differentiable with

$$f'(x) = s'(x)\left(\int_x^1 \frac{s(1)-s(y)}{s(1)-s(0)}h(y)m(dy) \;-\; \int_0^x \frac{s(y)-s(0)}{s(1)-s(0)}h(y)m(dy)\right)$$

Hence $f'(x)/s'(x)$ is continuously differentiable with

$$Lf(x) = \frac{1}{m(x)} \frac{d}{dx} \left( \frac{f'(x)}{s'(x)} \right) = -h(x)$$

Then, by Assumption (iii), $f \in \mathcal{D}(A)$ and $Af = Lf = -h$. In fact, it is not difficult to show that if $f(x) \in C^2(I_0) \cap C_0(I)$ and $Lf = -h \in C_0(I)$, then $f(x)$ is given by (2.2). (***Exercise:*** Prove the last statement. Lemma 8.1 in Section 8 below has a second proof using martingales.)

**(3)** Another useful identity is

$$T_t f(x) = f(x) + \int_0^t T_s(Af)(x)ds, \quad \text{any } f \in \mathcal{D}(A) \tag{2.8}$$

This follows immediately from the definition (1.1) and the strong continuity of the semi-group.

**(4)** If $\|T_t\| \le Ce^{Mt}$ and

$$R_\lambda f(x) = \int_0^\infty e^{-\lambda t} T_t f(x) dt, \qquad \lambda > M \tag{2.9}$$

then applying $R_\lambda$ to the identity (2.8) and rearranging terms implies

$$R_\lambda \big(\lambda I - A\big) f = f, \qquad f \in \mathcal{D}(A)$$

Similarly, if $g = R_\lambda f$ for $f \in B$, it follows from the strong continuity of $\{T_t\}$ that $g \in \mathcal{D}(A)$ and $Ag = \lambda g - f$. Thus $(\lambda I - A)R_\lambda f = f$ for all $f \in B$. It follows that $\lambda I - A$ on $\mathcal{D} = \mathcal{D}(A)$ is a two-sided inverse of $R_\lambda$ on $B$.

**3. A Probability Space for $T_t$.** By Assumptions (i) and (ii), we can write

$$T_t f(x) = \int_0^1 f(y) P(t, x, dy) \tag{3.1}$$

where $P(t, x, dy) \ge 0$ are Borel measures on the open unit interval $I_0 = (0, 1)$. The semigroup property $T_{t+s} f = T_t T_s f$ for $\{T_t\}$ is equivalent to the *Chapman-Kolmogorov equations*

$$P(t + s, x, B) \ = \ \int_0^1 P(t, y, B) P(s, x, dy) \tag{3.2}$$

for $P(t, x, dy)$. The assumptions in Section 1 imply $0 \le P(t, x, I_0) \le 1$, but do not exclude the possibility $P(t, x, I_0) < 1$.

The first step in finding a stochastic process corresponding to the semigroup (3.1) is to extend $P(t,x,A)$ on $I_0=(0,1)$ to a larger space $I_\Delta$ such that (i) $P(t,x,I_\Delta)=1$ for all $x\in I_\Delta$ and (ii) the Chapman-Kolmogorov equations (3.2) hold on $I_\Delta$. To do this, we first define an abstract "death point" $\Delta\notin I_0$ and set

$$I_\Delta = I_0\cup\{\Delta\}, \qquad I_0=(0,1) \tag{3.3}$$

We extend $P(t,x,A)$ for $x\in I_0, A\subseteq I_0$ to $x\in I_\Delta, A\subseteq I_\Delta$ by

$$\begin{aligned} P(t,x,\{\Delta\}) &= 1-P(t,x,I_0) \\ P(t,\Delta,\{\Delta\}) &= 1, \qquad P(t,\Delta,I_0) \;=\; 0 \end{aligned} \tag{3.4}$$

Then the extended function $P(t,x,A)$ for $x\in I_\Delta$ and $A\subseteq I_\Delta$ satisfies (i) $P(t,x,A)\ge 0$, (ii) $P(t,x,I_\Delta)=1$ for all $x\in I_\Delta$, and (iii) the Chapman-Kolmogorov relations

$$P(t+s,x,B) \;=\; \int_{I_\Delta} P(t,y,B)P(s,x,dy) \tag{3.5}$$

for $x\in I_\Delta$ and $B\subseteq I_\Delta$. (***Exercise:*** Prove (i,ii,iii) using (3.4) and (3.2) on $I_0$.)

Let $Q=\{\,k/2^m : k\ge 0, m\ge 0\,\}$ be the set of nonnegative dyadic rationals. We will use the Kolmogorov Consistency Theorem (KCT) to construct a probability space $(\Omega,\mathcal{F},P_x)$ with random variables $\{\,X_q(\omega)\in I_\Delta : q\in Q\,\}$ that form a Markov process consistent with (3.3), (3.4), and (3.5). The subscript $x$ in $P_x$ is a parameter $x\in I_\Delta$ for which $P_x(X_0=x)=1$. In the Kolmogorov representation, the basic sample space is the infinite product

$$\Omega \;=\; (I_C)^Q \;=\; \{\,\omega : \omega=\omega(r)\in I_C,\ r\in Q\,\} \tag{3.6}$$

where $I_C=[0,1]\cup\{\Delta\}$ is a compact version of $I_\Delta$. The random variables $X_r(\omega)=\omega(r)$ are the coordinate functions in the infinite product $\Omega$. The sigma-algebra $\mathcal{F}=\mathcal{B}\{\,X_r : r\in Q\,\}$ is the smallest sigma-algebra of subsets of $\Omega$ with respect to which all $X_r$ are measurable. This is also the sigma algebra generated by the *cylinder sets*

$$\Gamma(r_1,\ldots,r_n,A_1,\ldots,A_n) \;=\; \{\,\omega : X_{r_i}(\omega)\in A_i \text{ for } 1\le i\le n\,\} \tag{3.7}$$

for $0\le r_1<r_2<\ldots<r_n$ for $r_i\in Q$, sets $A_i\subseteq I_C$, and all $n$. The first step in applying the KCT is to define $P_x$ for cylinder sets. The appropriate

definition in this case turns out to be the somewhat complex

$$\begin{aligned} &P_x\Big(\Gamma(r_1,\dots,r_n,A_1,\dots,A_n)\Big) \\ &= \int_{A_1}\dots\int_{A_n} P(r_1,x,dy_1)P(r_2-r_1,y_1,dy_2)\dots \\ &\qquad\qquad \dots\quad P(r_n-r_{n-1},y_{n-1},dy_n \end{aligned} \tag{3.8}$$

We assume in (3.8) that $P_x(X_0 = x) = 1$ if $r_1 = 0$, $P(t,0,\{0\}) = P(t,1,\{1\}) = 1$ for $t \ge 0$, and $P(t,x,\{0,1\}) = 0$ for $x \in I_\Delta = (0,1)\cup\{\Delta\}$.

The next step is to notice that if (for example) $A_{i_0} = I_C$ in (3.7), then $P_x(\Gamma(r_1,\dots,A_1,\dots)$ is the same as if we dropped the $i_0$-th coordinate in (3.7) and (3.8) and computed the probability of the resulting cylinder set with $n-1$ conditions. (This is the *consistency condition* in the KCT.) Here the consistency condition follows from the Chapman-Kolmogorov relation

$$\int_{I_C} P(r_{i_0+1}-r_{i_0},y,A)P(r_{i_0}-r_{i_0-1},x,dy) \;=\; P(r_{i_0+1}-r_{i_0-1},x,A)$$

which follows from (3.5).

The KCT then implies that there exists a unique probability measure $P_x$ on $(\Omega,\mathcal{F})$ such that $P_x\big(\Gamma(\dots)\big)$ is given by (3.8) for cylinder sets (3.7). One can then check from (3.8) that

$$\begin{aligned} P_x(X_r\in I_\Delta) &\;=\; 1, \qquad x\in I_\Delta\subset I_C,\ r\in Q \\ P_x(X_r\in A) &\;=\; P(r,x,A), \qquad x\in I_\Delta,\ A\subseteq I_\Delta \quad\text{and} \\ P_x(X_q\in A\mid \mathcal{B}_r)(\omega) &\;=\; P(q-r,X_r(\omega),A) \end{aligned} \tag{3.9}$$

for $r,q \in Q$, $r < q$, and $x \in I_\Delta$. Here $\mathcal{B}_r = \mathcal{B}\{\,X_a : a \le r,\, a\in Q\,\}$ is the smallest $\sigma$-algebra $\mathcal{B}_r \subseteq \mathcal{F}$ such that $X_a(\omega)$ is $\mathcal{B}_r$-measurable for $a \le r, a \in Q$. Here $P_x(X_q \in A \mid \mathcal{B}_r)(\omega)$ is the conditional-probability random variable. That is, $g(\omega) = P_x(X_q \in A \mid \mathcal{B}_r)(\omega)$ is the unique bounded $\mathcal{B}_r$-measurable function such that

$$E\big(I_A(X_q)h\big) = E(\big(P(q-r,X_r,A)h\big)$$

for all bounded $\mathcal{B}_r$-measurable random variables $h(\omega)$. (***Exercise:*** Verify (3.9).)

The relations (3.9) imply that, for a set of $\omega\in\Omega$ with $P_x$-probability one for every $x \in I_\Delta = (0,1)\cup\{\Delta\}$, the sample paths $\{\,X_r(\omega) : r\in Q\,\}$ start in and remain in $I_\Delta$. Thus $\{\, X_r : r \in Q \,\}$ is a Markov stochastic process with values in $I_\Delta$ and Markov transition function $P(t,x,A)$.

Similarly, (3.4) implies that, again for a set of $\omega$ with $P_x$-probability one for every $x \in I_\Delta$, $X_r(\omega) = \Delta$ for any $\omega$ implies $X_q(\omega) = \Delta$ for every $q > r$. Set $\zeta(\omega) = \min\{\, r \in Q : X_r(\omega) = \Delta \,\}$. We call $\zeta(\omega)$ the *death time* for a sample path $X_r(\omega)$ in $I_0$. For these $\omega \in \Omega$, $X(r,\omega) = \Delta$ whenever $r > \zeta(\omega)$. Thus, within sets of probability zero,

$$\{\, \omega : \zeta(\omega) > r \,\} \;=\; \{\, \omega : X_r(\omega) \in I_0 \,\}$$

This implies that by (3.1) and (3.3)–(3.4)

$$\begin{aligned} T_r f(x) \;&=\; \int_0^1 f(y) P(r,x,dy) \;=\; E_x\big(f(X_r) I_{[X_r \in I_0]}\big) \\ &=\; E_x\big(f(X_r) I_{[\zeta > r]}\big) \end{aligned} \tag{3.10}$$

The purpose for restricting $r$ to a countable set is so that events involving the sample paths $X_r(\omega)$ depend on at most a countably infinite number of random variables. Otherwise, simple events involving sample-path continuity or even Lebesgue measurability may not be measureable.

The next step will be to extend $\{\, X_r : r \in Q \,\}$ to random variables $\{\, X_t : t \geq 0 \,\}$ defined on the same probability space in such a way that events involving the process $\{\, X_t : t \geq 0 \,\}$ will depend only on $\{\, X_r : r \in Q \,\}$.

**4. Sample Path Regularity.** As in (2.2),

$$\begin{aligned} f(x) &= \int_0^1 g(x,y) h(y) m(dy) \qquad (4.1) \\ &= \big(1 - s(x)\big) \int_0^x s(y) h(y) m(dy) \;+\; s(x) \int_x^1 (1 - s(y)) h(y) m(dy) \end{aligned}$$

for $h \in C_0(I)$. Define functions $\phi_N(x) \in C_0(I)$ for $N \geq 3$ such that $\phi_N(x) = 0$ for $x < 1 - 1/N$, $\phi_N(x) \geq 0$,

$$\mathrm{Supp}(\phi_N) \subseteq \Big(1 - \frac{1}{N},\, 1 - \frac{1}{2N}\Big)$$

and $\int_0^1 \big(1 - s(y)\big) \phi_N(y) m(dy) = 1$. Define

$$f_N(x) = \int_0^1 g(x,y) \phi_N(y) m(dy) \tag{4.2}$$

Then $f_N \in \mathcal{D}(A)$, $Af_N(x) = -\phi_N(x) \leq 0$, and by (4.1)

$$f_N(x) = s(x) \quad \text{on} \quad \left(0,\, 1 - \frac{1}{N}\right) \tag{4.3}$$

It follows from the identity (2.8) in Section 2 that

$$\begin{aligned} T_t f_N(x) &= f_N(x) + \int_0^t T_s(Af_N)ds \\ &= f_N(x) - \int_0^t T_s \phi_N(x)ds \end{aligned} \tag{4.4}$$

Since $\phi_N(x) \geq 0$ and $f_N(x) \geq 0$ by (4.2), $0 \leq T_t f_N(x) \leq f_N(x)$ for all $t \geq 0$ and $x \in I_0$. Then by (3.10), (4.4), and the Markov property

$$\begin{aligned} E_x\Big(f_N(X_q)I_{[\zeta>q]} \,\Big|\, \mathcal{B}_r\Big)(\omega) &= T_{q-r}f_N\big(X_r(\omega)\big)) \\ &\leq f_N\big(X_r(\omega)\big)I_{[\zeta(\omega)>r]} \end{aligned} \tag{4.5}$$

where $\mathcal{B}_r = \mathcal{B}\{\, X_a : a \leq r,\, a \in Q \,\}$ as before. Since $0 \leq f_N(x) \leq C_N$ by (4.2) and (2.7),

$$Y_r^N(\omega) = f_N\big(X_r(\omega)\big)I_{[\zeta(\omega)>r]}, \qquad 0 \leq r < \infty,\ r \in Q$$

is a uniformly bounded supermartingale for each $N$. Moreover, it follows from (4.3) and the conditional Fatou's inequality

$$E\Big(\liminf_{N\to\infty} H_N \,\Big|\, \mathcal{B}_r\Big) \;\leq\; \liminf_{N\to\infty} E\Big(H_N \,\Big|\, \mathcal{B}_r\Big) \text{ a.s.,}$$

for arbitrary random variables $H_N(\omega) \geq 0$ that (4.5) also holds with $f_N(x)$ replaced by $s(x)$. Thus

$$Y_r(\omega) = s\big(X_r(\omega)\big)I_{[\zeta(\omega)>r]} \qquad 0 \leq r < \infty,\ r \in Q \tag{4.6}$$

is also a uniformly bounded supermartingale.

Doob's Upcrossing Inequality now applies to the finite subsets $Q_{NM} = \{\, k/2^N : 0 \leq k \leq M \,\}$ of $Q$ with a uniform upper bound. This in turn implies that there exists a single null set $E \in \mathcal{F}$ such that for $\omega \notin E$, the limits

$$\lim_{q>t,\, q\downarrow t} X_q(\omega) = X_{t+}(\omega), \qquad \lim_{q<t,\, q\uparrow t} X_q(\omega) = X_{t-}(\omega) \tag{4.7}$$

exist for all real values $t < \zeta(\omega)$, where the first limit exists for $t = 0$, and the second limit exists at $t = \zeta(\omega)$. It follows from (4.7) that the set

$$\{\, t : t < \zeta(\omega),\ X_{t+}(\omega) \neq X_{t-}(\omega) \,\}$$

is at most countably infinite for each such $\omega$, even though the set of possible $t$ in (4.7) is uncountably infinite. Since $X_r(\omega) = \Delta$ almost surely for $r \ge \zeta(\omega)$, it follows that (4.7) holds for all real $t \ge 0$ and $\omega \notin E_1$ for a larger null set $E_1$.

We now define random variables $X_t(\omega)$ for all real $t \ge 0$ in terms of the random variables $\{\, X_r(\omega) : r \in Q \,\}$ by

$$\begin{aligned} X_t(\omega) \ &= \lim_{r>t,\, r\downarrow t} X_r(\omega) \ = \ X_{t+}(\omega), \qquad 0 \le t < \zeta(\omega) \\ &= \ \Delta, \qquad \zeta(\omega) \le t < \infty \end{aligned} \tag{4.8}$$

Note that this defines an uncountable number of random variables $\{X_t(\omega)\}$ in terms of a countable set $\{\, X_r(\omega) : r \in Q \,\}$. The first relation in (4.7) implies that $X_t(\omega)$ is right-continuous in $t$ for all real $t \ge 0$. The relations (4.7) and (4.8) do not exclude $X_t(\omega) \ne X_r(\omega)$ for $t = r$. However, the strong continuity condition on $T_f(x)$ in Section 1 implies

$$\lim_{q>r,\, q\downarrow r} T_q f(x) = \lim_{q>r,\, q\downarrow r} E_x\big(f(X_q) I_{[\zeta>q]}\big) = T_r f(x) = E_x\big(f(X_r) I_{[\zeta>r]}\big)$$

uniformly in $x \in I_0$ for all $f \in B = C_0(I)$. This implies $\lim_{q>r,\, q\downarrow r} X_q = X_r$ weakly $P_x$ for all $x \in (0,1)$. Since the same limit converges a.s. to $X_t$ by (4.7), it follows that $P_x(X_t = X_r) = 1$ for $t = r$. Thus, with probability one, the sample paths $\{\, X_t(\omega) : t \ge 0 \,\}$ are right continuous in $t$. We will use another martingale argument below to show that, in fact, almost every sample path $\{X_t(\omega)\}$ is continuous for $t < \zeta(\omega)$.

A subtlety of the definition (4.8) is that $X_t(\omega)$ is not $\mathcal{B}_t$-measurable for $\mathcal{B}_t = \mathcal{B}\{\, X_r : r \le t \,\}$ even if $t = r \in Q$, since it involves a right-hand limit. However, $X_t$ is $\mathcal{B}_{t+}$ measurable for $\mathcal{B}_{t+} = \cap_{\epsilon>0} \mathcal{B}_{t+\epsilon}$. The right-continuity of sample paths from (4.8) implies that $X_t$ is a Markov process with respect to $\mathcal{B}_{t+}$: That is,

$$P_x(X_{t+s} \in A \mid \mathcal{B}_{t+}) \ = \ P(s, X_t(\omega), A) \text{ a.s.} \tag{4.9}$$

This follows from the relation

$$E_x\big(\phi(X_{t+s})\psi(X_{t_1}, \ldots, X_{t_n})\big) \ = \ E_x\big(T_{s-\epsilon}\phi(X_{t_n})\psi(X_{t_1}, \ldots, X_{t_n})\big) \tag{4.10}$$

for all $\phi \in C_0(I)$, $\psi \in C_0(I^n)$, and $0 < t_1 < t_2 < \ldots < t_n = t + \epsilon < t + s$. The relation (4.10) follows from (3.5) and (3.8). Finally, let $\epsilon \to 0$ in (4.10) and use the strong continuity of $\{T_t\}$ and the right-continuity of sample paths $X_t(\omega)$. (***Exercise:*** Carry out the details.)

**5. A Family of Martingales.** The key step in the last step was identifying a bounded supermartingale. Most of the following results will also depend on martingale arguments. Before continuing, we state a result about a set of martingales that we will use several times that arises naturally for any Markov process.

To keep things simple, first consider a discrete-time Markov proces $X_n$ on a state space $J$ with a transition function

$$P(X_{n+1} \in B \mid X_n = x) = \pi(x, B)$$

Since $X_n$ is a Markov process,

$$P(X_{n+1} \in B \mid \mathcal{B}(X_1, X_2, \ldots, X_n)) = \pi\big(X_n(\omega), B\big) \tag{5.1}$$

Define $Tf(x) = \int_J f(y)\pi(x, dy) = E_x\big(f(X_1)\big)$ and $Af(x) = Tf(x) - f(x)$. Then $T^n f(x) = E_x\big(f(X_n)\big)$ by the Markov property, and

**Lemma 5.1.** For any bounded function $f(x)$ on $J$, the process

$$Y_n = f(X_n) - \sum_{k=0}^{n-1} Af(X_k) \tag{5.2}$$

is a martingale with respect to the $\sigma$-algebras $\mathcal{B}_n = \mathcal{B}(X_1, X_2, \ldots, X_n)$.

**Proof.** Assume $n < m$. Clearly $\mathcal{B}_n \subseteq \mathcal{B}_m$ and $Y_n$ is $\mathcal{B}_n$-measurable. Thus it only remains to prove $E(Y_m \mid \mathcal{B}_n) = Y_n$ in order to show that $\{\, Y_n, \mathcal{B}_n \,\}$ is a martingale. By (5.2)

$$\begin{aligned} Y_m \;&=\; f(X_m) \;-\; \sum_{k=n}^{m-1} Af(X_k) \;-\; \sum_{k=0}^{n-1} Af(X_k) \\ &=\; Y_m^{(1)} \;-\; Y_m^{(2)} \;-\; Y_m^{(3)} \end{aligned}$$

Then

$$\begin{aligned} E(Y_m^{(1)} \mid \mathcal{B}_n) \;&=\; E\big(f(X_n) \mid \mathcal{B}_n\big) \;=\; T_{m-n} f(X_n) \\ E(Y_m^{(2)} \mid \mathcal{B}_n) \;&=\; \sum_{k=n}^{n-1} E\big(Af(X_k) \mid \mathcal{B}_n\big) \;=\; \sum_{k=n}^{n-1} T_{k-n} Af(X_n) \end{aligned}$$

by (5.1) and the Markov property for $X_n$. Since $Af = Tf - f$,

$$E(Y_m^{(1)} \mid \mathcal{B}_n) - E(Y_m^{(2)} \mid \mathcal{B}_n) \;=\; T^{m-n} f(X_n) \;-\; \sum_{k=n}^{m-1} T^{k-n} Af(X_n)$$

$$= T^{m-n}f(X_n) \; - \; \sum_{k=n}^{m-1}\big(T^{k-n+1}f(X_n) - T^{k-n}f(X_n)\big)$$
$$= f(X_n)$$

and $E(Y_m^{(1)} - Y_m^{(2)} \mid \mathcal{B}_n) = f(X_n)$. Since $Y_m^{(3)} = \sum_{k=0}^{n-1} Af(X_k)$ is $\mathcal{B}_n$-measurable,

$$\begin{aligned} E(Y_m \mid \mathcal{B}_n) \; &= \; E(Y_m^{(1)} - Y_m^{(2)} \mid \mathcal{B}_n) - E(Y_m^{(3)} \mid \mathcal{B}_n) \\ &= \; f(X_n) \; - \; \sum_{k=0}^{n-1} g(X_k) \; = \; Y_n \end{aligned}$$

This completes the proof that $\{ Y_n, \mathcal{B}_n \}$ is a martingale.

**Lemma 5.2.** For any function $f \in \mathcal{D}(A)$, the process

$$Y_t(\omega) \; = \; f\big(X_t(\omega)\big) I_{[\zeta(\omega)>t]} \; - \; \int_0^{t\wedge\zeta(\omega)} Af\big(X_s(\omega)\big)ds \tag{5.3}$$

is a martingale with respect to the sigma-algebras $\{\mathcal{B}_{t+}\}$.

**Proof.** By (2.8)

$$T_t f(x) \; = \; f(x) \; + \; \int_0^t T_s Af(x)ds \tag{5.4}$$

for any $f \in \mathcal{D}(A)$. Since $\Delta$ is a trap, any $f \in \mathcal{D}(A)$ for the semigroup $T_t f(x)$ for $f(x)$ on $I_0 = (0,1)$ can be extended to $f \in \mathcal{D}(A)$ for $f(x)$ on $I_\Delta = (0,1) \cup \{\Delta\}$ by setting $f(\Delta) = 0$. By essentially the same argument as in the proof of Lemma 5.1 with (5.4) replacing a telescoping sum, this implies that the process

$$Y_t \; = \; f(X_t) - \int_0^t Af(X_s)ds \tag{5.5}$$

is a continuous-parameter martingale with respect to $\mathcal{B}_{t+}$. Since $\Delta$ is a trap, $Af(\Delta) = 0$ for any $f \in \mathcal{D}(A)$. Thus the martingale (5.5) is the same as the process in (5.3).

**Exercise.** Complete the proof that (5.5) is a continuous-time martingale with respect to $\mathcal{B}_{t+}$. Give justifications for interchanging the order of integrals and conditional expectations.

**6. An Upper Bound for Exit and Death Times.** Let

$$T_N(\omega) = \min\Big\{\ t : t < \zeta(\omega),\ X_t(\omega) \in \Big(0,\ \frac{1}{N}\Big) \cup \Big(1 - \frac{1}{N},\ 1\Big)\ \Big\} \qquad (6.1)$$

be the first time that $X_t(\omega)$ exits from the closed interval $[1/N, 1-1/N]$, assuming that it remains within $I_0 = (0,1)$.

We make the the convention that $T_N(\omega) = \infty$ if the right-hand side of (6.1) is empty; that is, if $X_t(\omega) \notin (0, 1/N) \cup (1-1/N, 1)$ for all $t < \zeta(\omega)$. If $T_N(\omega) < \infty$, then $T_N(\omega) < \zeta(\omega)$.

Note that $\{\, \omega : T_N(\omega) > t\,\} \in \mathcal{B}_{t+}$. We may not have $\{\, \omega : T_N(\omega) > t\,\} \in \mathcal{B}_t$ in general, either because $t \notin Q$ (recall $\mathcal{B}_t = \mathcal{B}\{\, X_q : q \le t,\, q \in Q\,\}$) or because $t = q \in Q$ but $X_t = X_{q+} \ne X_q$. Thus $T_N$ may not be a $\mathcal{B}_t$-stopping time, but is a $\mathcal{B}_{t+}$-stopping time. However, Lemma 5.2 implies that $Y_t(\omega)$ are martingales with respect to $\mathcal{B}_{t+}$, so that this is enough.

We then have

**Theorem 6.1.** If $C$ is the constant in (2.7),

$$\max_{0 \le x \le 1} E_x(T_\infty \wedge \zeta)\ \le\ C, \qquad T_\infty(\omega) = \sup_N T_N(\omega) \qquad (6.2)$$

where $T_\infty \wedge \zeta = \min\{T_\infty, \zeta\}$.

Thus, with $P_x$-probability one for any $x \in I_0$, either $T_\infty(\omega) < \infty$ or $\zeta(\omega) < \infty$. In particular, the only way that $T_\infty(\omega) = \infty$ can happen is if $\zeta(\omega) < \infty$.

**Proof of Theorem 6.1.** Define continuous functions $\psi_N \in C_0(I)$ such that $0 \le \psi_N(y) \le 1$ and $\psi_N(y) = 1$ on $(1/N,\ 1-1/N)$. If

$$h_N(x) = \int_0^1 g(x,y)\psi_N(y)m(dy) \qquad (6.3)$$

then $h_N \in \mathcal{D}(A)$ and $Ah_N = -\psi_N \le 0$. In particular, $Ah_N(x) = -1$ on $(1/N,\ 1-1/N)$. Thus by Lemma 5.2

$$Y_t^N\ =\ h_N(X_t)I_{[\zeta > t]}\ +\ \int_0^{t\wedge\zeta} \psi_N(X_s)ds \qquad (6.4)$$

is a nonnegative martingale for any $N \ge 3$. Then by the Optional Stopping Theorem,

$$\begin{aligned} Y^N_{t\wedge T_N}\ &=\ h_N(X_{t\wedge T_N})I_{[\zeta > t\wedge T_N]}\ +\ \int_0^{t\wedge T_N\wedge\zeta} \psi_N(X_s)ds \\ &=\ h_N(X_{t\wedge T_N})I_{[\zeta > t\wedge T_N]}\ +\ t\wedge T_N \wedge \zeta \end{aligned}$$

are also martingales, since $\psi(X_s) = 1$ for $0 \le s < T_N \wedge \zeta$. This in turn implies

$$E_x(t \wedge T_N \wedge \zeta) \;=\; h_N(x) - E_x\big(h_N(X_{t\wedge T_N \wedge \zeta})\big) \;\le\; C \tag{6.5}$$

since $0 \le \psi_N(x) \le 1$ implies $0 \le h_N(x) \le C$ by (6.3) where $C$ is the same constant as in (2.7). Fatou's theorem applied to $t$ and $N$ (in any order) then implies (6.2).

**7. Exit Times are Less Than Death Times.** Death times $\zeta(\omega)$ are necessary for general Markov processes, but can be shown to be less mysterious here. As in (6.1), let

$$T_N(\omega) = \min\Big\{\; t : t < \zeta(\omega),\; X_t(\omega) \in \Big(0,\, \frac{1}{N}\Big) \cup \Big(1 - \frac{1}{N},\, 1\Big)\;\Big\} \tag{7.1}$$

We show below that $P_x(T_N < \zeta) = 1$, so that $T_N(\omega) < \zeta(\omega)$ almost surely. This implies that $T_\infty(\omega) = \sup_N T_N(\omega) \le \zeta(\omega)$ with probability one. We will show in a later section that, in fact, $T_\infty(\omega) = \zeta(\omega)$ almost surely.

The same argument as in Theorem 7.1 can also be used to show that the sample paths $\{X_t(\omega)\}$ are continuous for $0 \le t < \zeta(\omega)$ (see Theorem 7.2).

**Theorem 7.1.** For the process $\{\, X_t \,\}$ defined above,

$$P_x(T_N < \zeta) = 1 \quad \text{for all } x \in (0,1) \tag{7.2}$$

**Corollary 7.1.**

$$\max_{0\le x\le 1} E_x(T_\infty) \;\le\; C, \qquad T_\infty(\omega) = \sup_N T_N(\omega) \tag{7.3}$$

Morever, $\lim_{N\to\infty} X_{T_N}(\omega) = A$ exists almost surely where $A$ (depending on $\omega$) is either 0 or 1.

**Proof.** Equations (6.2) and (7.2) imply (7.3), $\lim_{N\to\infty} X_{T_N}$ exists a.s. by (4.7), and $X_{T_N} \in (0,1/N)\cup(1-1/N,1)$ by right-continuity of the sample paths.

**Proof of Theorem 7.1.** It is sufficient to assume $1/N < x < 1 - 1/N$ since $T_N(\omega) = 0$ if $x \in (0,1/N) \cup (1-1/N,1)$. It follows from (3.10) and the strong continuity of the semi-group $\{T_t\}$ that $P_x(\zeta > 0) = 1$ for arbitrary $x \in I_0$.

Let $\psi_N(x) \in C_0(I) \cap C^2(I)$ such that $0 \le \psi_N(x) \le 1$, $\psi_N(x) = 1$ on $(1/N, 1-1/N)$, and $0 \le \psi(x) < 1$ on $(0,1/N) \cup (1-1/N,1)$. (See the Exercise below.) Then $\psi_N \in \mathcal{D}(A)$ with $A\psi_N(x) = L\psi_N(x)$. By Lemma 5.2

$$Y_t^N(\omega) \;=\; \psi_N\big(X_t(\omega)\big)I_{[\zeta(\omega)>t]} \;-\; \int_0^{t\wedge\zeta(\omega)} L\psi_N\big(X_s(\omega)\big)ds \tag{7.4}$$

is a martingale. By the Optional Stopping Theorem

$$E_x\Big(Y^N_{t\wedge T_N}\Big) \;=\; E_x\Big(\psi_N\big(X_{t\wedge T_N}(\omega)\big)I_{[\zeta(\omega)>t\wedge T_N]}\Big) \;=\; \psi_N(x) \tag{7.5}$$

since $A\psi_N(X_s) = L\psi_N(X_s) = 0$ for $0 \le s < t\wedge T_N$.

Let $q_N(t,\omega)$ be the integrand in (7.5). Note that $0 \le t\wedge T_N \le T_N$. If $T_N < \zeta$, then $\lim_{t\to\infty} q_N(t,\omega) = \widetilde{\psi}_N = \psi_N(X_{T_N})$. If $T_N \ge \zeta$, then $\zeta(\omega) < \infty$ by (6.2) and $\lim_{t\to\infty} q_N(t,\omega) = 0$. In either case

$$\psi_N(x) \;=\; E_x\Big(\widetilde{\psi}_N I_{[\zeta(\omega)>T_N]}\Big) \tag{7.6}$$

where $\widetilde{\psi}_N = 0$ (for example) if $T_N \ge \zeta$. If $1/N < x < 1-1/N$, then the expected value $\psi_N(x) = 1$. Since the integrand of (7.6) is bounded by one, it must be equal to one a.s. This implies $P_x(\zeta(\omega) > T_N) = 1$, which completes the proof of Theorem 7.1.

**Exercise.** Let $k_N(x) = Nk(Nx)$ for

$$k(x) \;=\; \begin{cases} C\exp\big(-1/(1-x^2)\big) & \text{for } |x| < 1 \\ 0 & \text{for } |x| \ge 1 \end{cases}$$

where $C$ is chosen so that $\int_{-\infty}^{\infty} k(y)dy = 1$. Prove that (i) $k(x)$ has continuous derivatives of all orders on the real line $R$ (that is, $k \in C^\infty(R)$) and (ii) $\mathrm{Supp}(k) = [-1,1]$. Now set

$$\psi_N(x) \;=\; \int_{1/2N}^{1-1/2N} k_{2N}(x-y)dy$$

Prove that (iii) $\psi_N(x)$ has continuous derivatives of all orders on $R$, (iv) $0 \le \psi_N(x) \le 1$ for all $x$, (v) $\psi_N(x) = 1$ for $1/N \le x \le 1-1/N$, and (vi) $\psi_N(x) = 0$ if $x \le 0$ or $x \ge 1$.

**Remark.** It follows from (7.3) that

$$\max_{0\le x\le 1} P_x(T_\infty \ge t_0) \;\le\; 1/2, \qquad t_0 = 2C \tag{7.7}$$

This implies by the Markov property and standard arguments that

**Corollary 7.2.** Under the assumptions of Theorem 7.1

$$\max_{0\le x\le 1} P_x(T_\infty \ge t) \;\le\; 2e^{-\alpha t}, \quad t \ge 0 \tag{7.8}$$

for $\alpha = 2C\log 2 > 0$, and also

$$\max_{0\le x\le 1} E_x(e^{aT_\infty}) \ \le\ C_1(a), \quad 0\le a<\alpha \tag{7.9}$$

where $C_1(a)<\infty$ for $0\le a<\alpha$.

We can use the same argument as in Theorem 7.1 to prove

**Theorem 7.2.** With probability one, the sample paths $X_t(\omega)$ are continuous functions of $t$ for $0\le t<\zeta(\omega)$.

**Proof.** Since $P_x(T_N<\zeta)=1$ and $E_x(\zeta)\le C$, it follows from (7.6) that $\widetilde{\psi}_N=\psi_N(X_{T_N})=1$ a.s. in (7.6) for $1/N<x<1-1/N$. This implies that $X_{T_N}(\omega)\in\{1/N,1-1/N\}$ almost surely. This means that $X_t(\omega)$ cannot leave $(1/N,1-1/N)$ by a jump beyond the boundary of $(1/N,1-1/N)$. This suggests that, at least, $X_t(\omega)$ cannot be discontinuous at $X_t(\omega)=1/N$ or $X_t(\omega)=1-1/N$.

If general, define

$$T_{ab}(\omega)=\min\{\ t: t<\zeta(\omega),\ X_t(\omega)\in(0,a)\cup(b,1)\,\} \tag{7.10}$$

for rational values $a,b$ with $0<a<b<1$ with, as before, the convention $T_{ab}(\omega)=\infty$ if the set of $t$ on the right-hand side of (7.10) is empty. By the same argument as in Theorem 7.1, $P_x(T_{ab}<\infty)=1$ and $P_x(X_{T_{ab}}\in\{a,b\})=1$ for $0<a<x<b<1$.

Expand the null set $E\in\mathcal{F}$ in (4.7) to include the null set of all $\omega\in\Omega$ that provide counterexamples to $X_{T_{ab}(\omega)}(\omega)\in\{a,b\}$ for rational $0<a<b<1$. It follows that, if $\omega\notin E$, there cannot exist any rational values $r$ such that $X_{t-}(\omega)<r<X_{t+}(\omega)$, since we could then find a counterexample to $X_{T_{ar}(\omega)}(\omega)\in\{a,r\}$ for some rational interval $(a,r)$.

Similarly there cannot exist any rational values $r$ such that $X_{t+}(\omega)<r<X_{t-}(\omega)$ since we could find a counterexample to $X_{T_{rb}(\omega)}(\omega)\in\{r,b\}$ for some rational interval $(r,b)$. Since $X_t(\omega)$ has at worst jump discontinuities for $\omega\notin E$ by the argument that led up to (4.7), this implies that $X_t(\omega)$ must be continuous in $t$ for $0\le t<\zeta(\omega)$. This completes the proof of Theorem 7.2.

**8. Exit Times are Death Times.** The purpose of this section is to prove that, in fact, $T_\infty(\omega)=\zeta(\omega)$ almost surely for $T_\infty(\omega)=\sup_N T_N(\omega)$. We begin with two lemmas.

**Lemma 8.1.** If $f,h\in C_0(I)$, then $h\in\mathcal{D}(A)$ and $Ah=-f$ if and only if

$$h(x)=\int_0^1 g(x,y)f(y)m(dy) \tag{8.1}$$

**Proof.** If $h(x)$ is given by (8.1), then $h \in \mathcal{D}(A)$ and $Ah = Lh = -f$ as in Section 1. Conversely, assume $h \in \mathcal{D}(A)$ and $Ah = -f$. Define

$$h_1(x) = \int_0^1 g(x,y)f(y)m(dy)$$

Then $h_1 \in \mathcal{D}(A)$ and $Ah_1 = Lh_1 = -f$. Let $g = h - h_1$. Then $g \in \mathcal{D}(A)$ and $Ag = Ah - Ah_1 = 0$.

If $g \in \mathcal{D}(A)$ and $Ag = 0$, it follows from Lemma 5.2 that $g(X_t)I_{[\zeta>t]}$ is a uniformly-bounded martingale. Thus

$$g(x) \;=\; E_x\big(g(X_t)I_{[\zeta>t]}\big)$$

By the relation $P_x(T_N < \zeta) = 1$ and the Optional Stopping Theorem,

$$g(x) \;=\; E_x\big(g(X_{t\wedge T_N})\big)$$

Since $P_x(T_N < \infty) = 1$ by Theorems 6.1 and 7.1 and $g(x)$ is bounded, we can let $t \to \infty$ and obtain

$$g(x) \;=\; E_x\big(g(X_{T_N})\big)$$

However, $\lim_{N\to\infty} g(X_{T_N}) = 0$ since $g \in C_0(I)$ by Corollary 7.1. This proves that $g(x) = 0$ for $0 < x < 1$. Since $g = h - h_1$, it follows that $h(x) = h_1(x) = \int g(x,y)f(y)m(dy)$. This completes the proof of the lemma.

**Lemma 8.2.** For any $f \in C_0(I)$ with $f(x) \ge 0$,

$$\int_0^1 g(x,y)f(y)m(dy) \;=\; \int_0^\infty T_s f(x)ds \;=\; E_x\Big(\int_0^{\zeta(\omega)} f(X_s)ds\Big) \tag{8.2}$$

**Proof.** Define

$$h_\lambda(x) = R_\lambda f(x) = \int_0^\infty e^{-\lambda s}T_s f(x)ds$$

As in Section 2, $h_\lambda(x) \in \mathcal{D}(A)$ and $(\lambda I - A)h_\lambda(x) = f(x)$. Thus $Ah_\lambda = \lambda h_\lambda - f = -(f - \lambda h_\lambda)$. It then follows from Lemma 8.1 that

$$\begin{aligned} h_\lambda(x) = \int_0^\infty e^{-\lambda s}T_s f(x)ds &= \int_0^1 g(x,y)\big(f(y) - \lambda h_\lambda(y)\big)m(dy) \qquad (8.3)\\ \le \int_0^1 g(x,y)f(y)m(dy) \;&\le\; C\sup_y f(y) \end{aligned}$$

since $f(x) \geq 0$ and $h_\lambda(y) \geq 0$, where $C$ is the constant in (2.7). Thus $h_\lambda(x) \leq C_1$ uniformly in $\lambda > 0$. Since then $\lambda h_\lambda(x) \leq C_1\lambda$, taking the limit $\lambda \to 0$ in (8.3) for $f(x) \geq 0$ implies

$$\begin{aligned} h_0(x) \ &= \ \int_0^\infty T_s f(x)ds \ = \ \int_0^1 g(x,y)f(y)m(dy) \\ &= \ E_x\Big(\int_0^\infty f(X_s)I_{[\zeta>s]}ds\Big) \ = \ E_x\Big(\int_0^{\zeta(\omega)} f(X_s)ds\Big) \end{aligned}$$

This implies (8.2).

**Theorem 8.1.** Define $T_N$, $T_\infty$, and $\zeta$ as before. Then $P_x(T_\infty = \zeta) = 1$ for all $x \in I_0$.

**Proof.** Let

$$h(x) = \int_0^1 g(x,y)\psi(y)m(dy) \tag{8.4}$$

for arbitrary $\psi \in C_0(I)$ with $\psi(y) \geq 0$. Then $Ah = -\psi$ and

$$Y_t(\omega) \ = \ h\big(X_t(\omega)\big)I_{[\zeta(\omega)>t]} \ + \ \int_0^{t\wedge\zeta(\omega)} \psi\big(X_s(\omega)\big)ds$$

is a nonnegative martingale by Lemma 5.2. Thus

$$E_x\Big(h(X_t)I_{[\zeta>t]} \ + \ \int_0^{t\wedge\zeta} \psi(X_s)ds\Big) \ = \ h(x)$$

for all $t \geq 0$ and $x \in I_0$. By the Optional Stopping Theorem

$$\begin{aligned} h(x) \ &= \ E_x\Big(h(X_{t\wedge T_N})I_{[\zeta>t\wedge T_N]} \ + \ \int_0^{t\wedge T_N\wedge\zeta} \psi(X_s)ds\Big) \\ &= \ E_x\Big(h(X_{t\wedge T_N}) \ + \ \int_0^{t\wedge T_N} \psi(X_s)ds\Big) \end{aligned}$$

since $P_x(T_N < \zeta) = 1$ by (7.2). Since $h(x)$ is bounded, $\psi(y) \geq 0$, and $T_N < \infty$ almost surely by (6.2), we can let $t \to \infty$ and conclude

$$h(x) \ = \ E_x\Big(h(X_{T_N}) \ + \ \int_0^{T_N} \psi(X_s)ds\Big) \tag{8.5}$$

As $N \to \infty$, $h(X_{T_N}) \to 0$ since $h \in C_0(I)$. Since $0 \leq T_N \uparrow T_\infty$ and $\psi(X_s) \geq 0$, we conclude from (8.5) that

$$h(x) \ = \ E_x\Big(\int_0^{T_\infty(\omega)} \psi(X_s)ds\Big) \tag{8.6}$$

In contrast, by Lemma 8.2 and (8.4)

$$h(x) \;=\; E_x\Big(\int_0^{\zeta(\omega)} \psi(X_s)ds\Big) \tag{8.7}$$

Recall $T_\infty(\omega) \le \zeta(\omega)$ almost surely by (7.2). Both integrals in in (8.6) and (8.7) are finite since $\psi(X_s) \ge 0$ and $h(x)$ is finite. We can then subtract (8.6) from (8.7) to conclude

$$E_x\Big(\int_{T_\infty(\omega)}^{\zeta(\omega)} \psi(X_s)ds\Big) \;=\; 0$$

This holds for all $\psi \in C_0(I)$ with $\psi(x) \ge 0$. We can approximate $\psi(x) \equiv 1$ by nonnegative functions $\psi_N \in C_0(I)$ by setting (for example) $\psi_0(x) = 2\min\{\, x, 1-x \,\}$ and $\psi_N(x) = \psi_0(x)^{1/N}$. Then $0 \le \psi_N(x) \le 1$ and $0 \le \psi_N(x) \uparrow 1$ as $N \to \infty$. This implies

$$E_x\Big(\zeta(\omega) - T_\infty(\omega)\Big) = 0, \qquad T_\infty(\omega) \le \zeta(\omega) \text{ a.s.} \tag{8.8}$$

Thus $\zeta(\omega) - T_\infty(\omega) = 0$ almost surely with respect to $P_x$ for all $x \in I_0$. This completes the proof of Theorem 8.1. (***Exercise:*** Prove or disprove: (8.8) also holds for $x = \Delta$. Use the precise definitions of $X_r(\omega)$, $X_t(\omega)$, and $T_N(\omega)$. Discuss.)

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