Martin Boundaries and Random Walks

Stanley A. Sawyer
Washington University, St. Louis, USA

1. An Overview

The first three sections give a quick overview of Martin boundary theory and state the main results. The succeeding sections will flesh out the details, and give proofs and examples.

Virtually all of the results below are classical. The article Doob (1959) and the book by Kemeny, Snell, and Knapp (1976) are good sources for additional details. A recent survey article by Wolfgang Woess (1994) has an immense amount of information (both modern and classical) about Martin boundaries and random walks in general. Finally, Doob (1984) is an excellent source for classical Martin boundary theory for Brownian motion and the Laplacian.

Introduction

Let S be a group or a homogeneous space of a group. Martin boundary theory can be used to do the following:

(i) Characterize all nonnegative harmonic functions on S, and sometimes also construct them,

(ii) Characterizes the behavior of random walks X_n on S as $n \to \infty$, and

(iii) Define and work with "h-processes" on S, which are processes like X_n but which can have simpler limiting behavior as $n \to \infty$. These can be used to give more information about harmonic functions on S and about the Martin boundary itself.

For simplicity, we assume that S is a countable discrete group or homogeneous space, in which case the random walks X_n are Markov chains. The same results below hold generally as well for Lie groups and symmetric spaces with X_n replaced by the intrinsic "Brownian motion" diffusion processes X_t on S. The main differences between the discrete and continuous cases are greater technicalities in the construction of the Martin boundary, some of which is caused by worries about the regularity of the sample paths of X_t. On the other hand, there are fewer Brownian motion processes X_t on a particular Lie group G than possible random walks X_n on a countable group, which makes the continuous case easier in that respect.

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Martin boundary theory can be applied to nearly any transient Markov chain or continuous-time Markov processes. Most of the conclusions below hold in some form whether X_n is a random walk or Brownian motion process or not.

For definiteness, assume that S is either a countably infinite group or else is a countable set that is acted upon transitively by a group G. Assume that we have an "averaging" or "transition" function $p(x,y) \ge 0$ on $S \times S$ such that

(i) $\sum_{y\in S} p(x,y) = 1$ for all $x \in S$, and $p(x,y) \ge 0$ for $x,y \in S$,

(ii) $p(x,y) = p(gx,gy)$ for all $x,y \in S$ and $g \in G$.

A function $u(x) \ge 0$ on S is called *harmonic* (or *p-harmonic*) if

$$u(x) = \sum_{y\in S} p(x,y)u(y) \qquad \text{for all } x \in S \tag{1.1}$$

The simplest case is when S is a graph and $p(x,y)$ puts equal weight on the nodes that are one link away from x. In this case, $p(x,y)$ is called the *isotropic nearest-neighbor* transition function on S. A function $u(x)$ on S is then harmonic if it is equal to the average of the values of $u(x)$ on the nearest neighbors of x in the graph. Two important special cases are when S is

(i) The d-dimensional lattice Z^d in R^d or

(ii) The infinite homogeneous tree T_r in which each element x has $r \ge 3$ nearest neighbors.

We will usually assume that $p(x,y)$ is both *irreducible* and *transient*. *Irreducibility* means that the associated random walk X_n can get from any $x \in S$ to any other $y \in S$ (with positive probability). *Transience* means that, with probability one, the associated random walk X_n eventually wanders off to infinity. (See below for a more precise definition.) If $p(x,y)$ is irreducible and has any nonconstant nonnegative harmonic functions, then it is automatically transient.

By Polya's theorem, isotropic nearest-neighbor random walk on Z^d is transient for $d \ge 3$ but not for $d = 1$ or $d = 2$. However, nonnegative nearest-neighbor harmonic functions on Z^d are constant for any d. Nearest-neighbor random walks on the infinite homogeneous tree T_r are transient, and there is a rich class of nonnegative harmonic functions.

If S is a group, the condition $p(x,y) = p(gx,gy)$ for all x,y,g is equivalent to

$$p(x,y) = p(e,x^{-1}y) = p_e(x^{-1}y), \qquad \text{all } x,y \in S$$

As mentioned above, there are no nonconstant isotropic nearest-neighbor nonnegative harmonic functions on $S = Z^d$, and the corresponding Martin boundary theory is trivial. The same holds more generally if $\sum_y p(e,y)|y| < \infty$ and $\sum_y p(e,y)y = 0$.

If $\sum_y p(e,y)y \ne 0$ in Z^d, there are generally nonconstant nonnegative harmonic functions for which there is a nice characterization in terms of Martin boundary theory that we will discuss in Section 7 below (Choquet and Deny, 1960; Doob, Snell, and Williamson, 1960; Ney and Spitzer, 1966; *see also* Woess, 1994). There is also a nice characterization of the Martin boundary and nonnegative isotropic nearest-neighbor harmonic functions on homogeneous trees that we present in Section 8, as well as a large literature for more general $p(x,y)$ on more general trees (Woess, 1994).

Section 6 below has a simple example in which Martin boundary theory for a random walk in Z^2 is used to solve a special case of the classical Hausdorff moment problem (Widder, 1946) and is also applied to an urn model.

The Martin Boundary

The basic approach is as follows (see Section 4 for the details). Given a transition function $p(x, y)$ which is transient and irreducible on a set S, we construct the *Martin compactification*

$$\widehat{S}_M = S \cup \partial S_M \qquad \text{of } S$$

(which depends on $p(x, y)$) and the *Martin boundary* $\partial S_M = \widehat{S}_M - S$. (More precisely, we construct a compact metric space $\widehat{S}_M$ with a homeomorphic embedding $\pi : S \to \widehat{S}_M$ and define $\partial S_M = \widehat{S}_M - \pi(S)$.)

As part of the same process, we obtain the *Martin kernel* $K(x, \alpha) \geq 0$ for $x \in S$ and $\alpha \in \partial S_M$ which has the property that, for any nonnegative p-harmonic function $u(x)$ on S, there exists a measure $\mu_u(d\alpha) \geq 0$ on ∂S_M such that

$$u(x) \;=\; \int_{\partial S_M} K(x, \alpha)\, \mu_u(d\alpha), \qquad \text{all } x \in S \tag{1.2}$$

(Here we follow the usual probabilist's convention of writing $\mu(d\alpha)$ instead of $d\mu(\alpha)$ for integrals with respect to a measure μ.)

It follows from the construction of $K(x, \alpha)$ and $\widehat{S}_M$ that

(i) $K(x_0, \alpha) = 1$, all $x \in \partial S_M$, for some $x_0 \in S$,

(ii) $K(x, \alpha) \leq C_x$ independently of $\alpha \in \partial S_M$

where $x_0 \in S$ is an arbitrary preassigned "reference point." This implies

(iii) $\mu_u(\partial S_M) = u(x_0) < \infty$, and

(iv) the integral in (1.2) converges whenever $\mu_u(\partial S_M) < \infty$

The natural question of whether the measure μ_u in (1.2) is unique, or whether $K(x, \alpha)$ is harmonic in x for all α, so that (1.2) defines a harmonic function for any probability measure μ_u on ∂S_M, leads to an additional technicality. The measures $\mu_u(d\alpha)$ in (1.2) are unique, and do define harmonic functions on S whenever $\mu_u(\partial S_M) < \infty$, providing that they are restricted to a subset $\partial_m S_M \subseteq \partial S_M$. In general, a function $u(x) \geq 0$ is called *minimal harmonic* on S if (i) it is harmonic and (ii) whenever $0 \leq w(x) \leq u(x)$ for any other harmonic function $w(x)$ on S, then $w(x) = Cu(x)$ for some constant $C \geq 0$. Set

$$\partial_m S_M = \{\, \alpha \in \partial S_M : K(x, \alpha) \text{ is minimal harmonic in } x \,\}$$

Then $\partial_m S_M$ is a Borel subset of ∂S_M, and, for any p-harmonic $u(x) \geq 0$ on S, there exists a *unique* measure $\mu_u(d\alpha) \geq 0$ with $\operatorname{Supp}(\mu_u) \subseteq \partial_m S_M$ such that

$$u(x) \;=\; \int_{\partial_m S_M} K(x, \alpha)\, \mu_u(d\alpha), \qquad x \in S \tag{1.3}$$

The set $\partial_m S_M \subseteq \partial S_M$ is called the "minimal boundary" of S. In many cases $\partial_m S_M = \partial S_M$ (see the examples below), but $\partial_m S_M \subset \partial S_M$ can occur even for random walks in Z^1 (Cartwright and Sawyer, 1991; Woess, 1994). Since $K(x, \alpha)$

is harmonic for each $\alpha \in \partial_m S_M$, the integral (1.3) defines a harmonic function $u(x) \geq 0$ for any measure $\mu_u(d\alpha) \geq 0$ with $\mu_u(\partial_m S_M) < \infty$.

The representations (1.2)-(1.3) hold for arbitrary nonnegative harmonic functions. Sometimes we have the representation

$$u(x) \;=\; \int_{\partial_P S_M} K(x,\alpha)\, \mu_u(d\alpha), \qquad x \in S \tag{1.4}$$

with a *smaller* subset $\partial_P S_M \subset \partial_m S_M$ for *bounded* nonnegative harmonic functions $u(x)$. The set $\partial_P S_M$ in (1.4) is called the *Poisson boundary* for $p(x,y)$. Typically $\partial_P S_M \subset \partial_m S_M$ for random walks in Z^d (see Section 7), but $\partial_P S_M = \partial_m S_M = \partial S_M$ for isotropic nearest-neighbor random walks in the infinite homogeneous tree T_r. In Section 6 we consider a random walk in Z^2 with $\partial_m S_M = \partial S_M = [0,1]$ (within homeomorphisms) but $\partial_P S_M = \{1/2\}$.

The uniqueness of the representing measure $\mu_u(d\alpha)$ in (1.3) can be used to find a general characterization of $\partial_P S_M$. If $0 \leq u(x) \leq w(x)$ are two p-harmonic functions, then $w(x) - u(x) \geq 0$ is also p-harmonic, so that $\mu_w(d\alpha) = \mu_u(d\alpha) + \mu_{w-u}(d\alpha)$ on $\partial_m S_M$ by uniqueness. Thus $\mu_u(d\alpha) \leq \mu_w(d\alpha)$ in (1.3) whenever $0 \leq u(x) \leq w(x)$. This implies that if $u(x)$ is a p-harmonic function with $0 \leq u(x) \leq M$, then $0 \leq \mu_u(d\alpha) \leq M\mu_1(d\alpha)$ where $\mu_1(d\alpha)$ is the representing measure of the constant p-harmonic function $u(x) \equiv 1$. That is, if $\mu_1(d\alpha)$ is defined by

$$\int_{\partial_m S_M} K(x,\alpha)\, \mu_1(d\alpha) \;=\; 1, \qquad \text{all } x \in S \tag{1.5}$$

then $\mu_u(d\alpha) \leq M\mu_1(d\alpha)$, and $u(x)$ is p-harmonic with $0 \leq u(x) \leq M$ if and only if

$$u(x) \;=\; \int_{\partial_m S_M} K(x,\alpha)\, g(\alpha)\, \mu_1(d\alpha) \tag{1.6}$$

where $0 \leq g(\alpha) \leq M$. Thus the Poisson boundary $\partial_P S_M$ is essentially the support of the representing measure μ_1 in (1.5).

Minimal Harmonic Functions and Choquet Theory

It follows from the uniqueness of the representation (1.3) that any minimal p-harmonic function $u(x)$ on S is of the form $CK(x,\alpha)$ for some constant C and $\alpha \in \partial_m S_M$. That is,

$$\{\, u_\alpha(x) = K(x,\alpha) : \ \alpha \in \partial_m S_M \,\}$$

is the entire set of minimal nonnegative p-harmonic functions on S within nonnegative constants.

The formula (1.3) is reminiscent of the Krein-Milman and Choquet theorems. The Krein-Milman theorem says that a compact convex set K in a linear topological space T is the closed convex hull of the extreme points of K. The Choquet theorem goes one step further and says that any $x \in K$ can be represented as the integral of a probability measure on the extreme points. If $p(x,y)$ is irreducible, the set

$$K_{x_0} \;=\; \{\, u(x) \geq 0 : \ u(x) = pu(x) \ \text{ for all } x, \quad u(x_0) = 1 \,\}$$

(where $pu(x) = \sum_{y \in S} p(x,y)u(y)$) is a compact convex set under the topology of pointwise convergence on S. Since S is discrete, this is the same topology as uniform convergence on compact sets in S. The extreme points of K_{x_0} are exactly the minimal harmonic functions, and (1.3) is equivalent to Choquet's theorem.

2. Random Walks and Probability Theory

The conditions

$$p(x,y) \geq 0, \qquad \sum_{y\in S} p(x,y) = 1 \qquad \text{for all } x \in S \tag{2.1}$$

are sufficient to construct a *Markov chain* on S with transition function $p(x,y)$. That is, given (2.1), there exists a measure space $(W,\mathcal{B})$ with

(i) probability measures P_x on $(W,\mathcal{B})$ depending on a parameter $x \in S$ (a *probability measure* is a nonnegative measure with total mass $P_x(W) = 1$) and

(ii) S-valued random variables $\{X_n(w) : n \geq 0\}$ for $w \in W$ (a random variable is the same as a measurable function) such that

$$\begin{aligned} P_x\big(\{\, w : X_0(w) = x,\ X_1(w) = y_1,\ X_2(w) = y_2,\ \ldots,\ X_n(w) = y_n \,\}\big) \\ = \ p(x,y_1)\, p(y_1,y_2)\ \ldots\ p(y_{n-1},y_n) \end{aligned} \tag{2.2}$$

for all $n \geq 0$ and $x,\ y_1, y_2, \ldots, y_n \in S$.

The notation "P_x" means that the process starts with $X_0 = x$. Note that $P_x(\{w : X_0(w) = x\}) = P_x(X_0 = x) = 1$ by summing (2.2) over $y_1, \ldots, y_n$ and applying (2.1). In general, P_x of a relation means P_x of the set of all $w \in W$ for which the relation is true. For example, $P_x(X_0 = x) = P_x\big(\{w : X(w) = x\}\big) = 1$. If $P_x(E) = 1$, the set E is said to happen *almost surely* (a.s.). That is, a.s. (almost surely) is the same as a.e. (almost everywhere).

An *event* in a probability space is the same as a measurable set $A \in \mathcal{B}$. The conditional probability of an event A given another event B is $P_x(A \mid B) = P_x(A \cap B)/P_x(B)$, which is defined whenever $P_x(B) > 0$. It follows from (2.2) that

$$P_x(X_{n+1} = y \mid X_n = x,\ X_1 = a_l, \ldots, X_{n-1} = a_{n-1}) \ = \ p(x,y)$$

for all $x, y \in S$ and $a_1, \ldots, a_{n-1} \in S$. This means that, once the process is at $X_n = x$, it forgets where it was before time n. (This is called the *Markov property* for the process $\{X_n\}$.)

The *expected value* $E_x(X)$ of a random variable $X(w)$ on $(W,\mathcal{B})$ is the integral $\int_W X(w)\, P_x(dw) = \int_W X(w)\, dP_x(w)$. Thus if $u(y) \geq 0$

$$\sum_{y\in S} p(x,y)u(y) \ = \ E_x\big(u(X_1)\big) \ = \ \int_W u\big(X_1(w)\big)\, P_x(dw)$$

and $u(x) \geq 0$ is p-harmonic if and only if $u(x) = E_x\big(u(X_1)\big)$ for all $x \in S$.

Random Walks

A Markov chain $\{\, X_n \,\}$ with transition function $p(x,y)$ on a set S is called a *random walk* with respect to a group G if G acts transitively on S and $p(x,y) = p(gx,gy)$ for all $x, y \in S$, $g \in G$. If $G = S$, this is equivalent to

$$p(x,y) = p(e, x^{-1}y) = p_e(x^{-1}y)$$

The S-valued random variables X, Y are *independent* if $P_x(X = a, Y = b) = P_x(X = a)P_x(Y = b)$ for all $a, b \in S$. This is equivalent to $P_x(X = a \mid Y = b) = P_x(X = a)$ if $P_x(Y = b) > 0$. If $p(x, y) = p(gx, gy)$ for $g \in G = S$, then X_n is a Markov chain with transition function $p(x, y)$ if and only if $\{X_n\}$ have the same joint probability distributions as

$$X_n = X_0 Y_1 Y_2 \ldots Y_n$$

where $\{ Y_n \}$ are independent S-valued random variables with the same distribution $P(Y_k = z) = p_e(z) = p(e, z)$.

Given a transition function $p(x, y)$ on a discrete set S, we can construct a probability space $(W, \mathcal{B})$ as follows. Let

$$W \;=\; S^\infty \;=\; \{\, w = (a_0, a_1, \ldots, a_n, \ldots) : \;\; a_i \in S \,\} \tag{2.3}$$

be the set of all infinite sequences from S. Define $X_n(w) = a_n$. For $n \ge 0$ and $y_i \in S$, the *cylinder set* $C = C_{n;y_0,\ldots,y_n} \subseteq W$ is the set of all $w \in W$ such that $w_i = y_i$ for $0 \le i \le n$. The *Borel σ-algebra* $\mathcal{B}$ is the σ-algebra generated by these cylinder sets. The relation (2.2) defines $P_x(C)$ on cylinder sets, and extends uniquely to a probability measure $P_x(B)$ on $(W, \mathcal{B})$ that satisfies (2.2).

For (2.3), the *sample paths* $\{X_n(w) : w \in W\}$ are exactly the same as the *points* $w = (a_0, a_1, \ldots, a_n, \ldots) \in W$. The probability space $(W, \mathcal{B})$ defined above is called the *Kolmogorov representation space* for the random variables X_n, and is a useful tool in measure-theoretic probability theory.

Random Walks and Transience

If we sum (2.2) over $y_1, y_2, \ldots, y_{n-1} \in S$, we obtain

$$P_x(X_n = y) \;=\; p_n(x, y), \qquad x, y \in S$$

where $p_n(x, y)$ is the n^{th} matrix power of $p(x, y)$. That is, $p_0(x, y) = \delta_{xy}$, $p_1(x, y) = p(x, y)$, and $p_n(x, y) = \sum_{z \in S} p_{n-1}(x, z)p(z, y)$ for $n \ge 1$. The *Green's function* or *potential function* of $p(x, y)$ or X_n is

$$\begin{aligned} g(x, y) \;&=\; \sum_{n=0}^{\infty} p_n(x, y) \\ &=\; \sum_{n=0}^{\infty} P_x(X_n = y) \;=\; E_x\Big(\sum_{n=0}^{\infty} I_y(X_n)\Big) \;\le\; \infty \end{aligned} \tag{2.4}$$

where $I_A(z)$ is the indicator function of the set A and $p_0(x, y) = I_x(y)$. Probabilistically, $g(x, y)$ is the expected number of times that the process is ever at y given that it began at $X_0 = x$.

The process X_n (or the kernel $p(x, y)$) is called *irreducible* if, for all $x, y \in S$, there exists some $n \ge 0$ such that $P_x(X_n = y) = p_n(x, y) > 0$. By (2.4), this is equivalent to $g(x, y) > 0$ for all $x, y \in S$. The process X_n is *transient* if

$$P_x(\lim_{n\to\infty} X_n = \infty) \;=\; 1, \qquad \text{all } x \in S \tag{2.5}$$

where $\lim_{n\to\infty} X_n = \infty$ means that, for any compact or finite set $K \subseteq S$, there exists $n_K < \infty$ such that $X_n \notin K$ for all $n \geq n_K$. In other words, X_n eventually leaves any finite set K and never returns. (This is the same as convergence to the point ∞ for the one-point compactification $\widehat{S} = S \cup \{\infty\}$ of S.)

One can show that X_n is transient if and only if $g(x,y) < \infty$ for all $x, y \in S$. If X_n is irreducible, then (2.5) holds for all $x \in S$ if it holds for one $x \in S$. Similarly, if (2.5) fails, then $g(x,y) = \infty$ for all $x, y \in S$, and also (since X_n is irreducible)

$$P_x(X_n = y \ \text{ for infinitely many } n \geq 0) \ = \ 1, \qquad \text{all } x, y \in S$$

In this case, $p(x,y)$ and X_n are called *recurrent*.

By a theorem of Polya, isotropic nearest-neighbor random walk in the d-dimensional lattice Z^d is transient for dimensions $d \geq 3$, but recurrent if $d = 1$ or $d = 2$. Nearest-neighbor random walk in the infinite homogeneous tree T_r ($r \geq 3$) is always transient.

3. Limits at Infinity

Assume that $\{X_n\}$ is transient and irreducible in S. Even though the sample paths $X_n(w)$ do not converge in S as $n \to \infty$, they converge a.s. in the Martin compactification $\widehat{S}_M$. (See Sections 4 and 5 for proofs.) Moreover, the limit $X_\infty(w) \in \partial_m S_M$ a.s. That is,

$$P_x(\lim_{n\to\infty} X_n \text{ exists and is some } X_\infty \in \partial_m S_M) \ = \ 1, \qquad \text{all } x \in S$$

We can also characterize the distribution of the exit point X_∞. Specifically

$$P_x(\lim_{n\to\infty} X_n = X_\infty \in A) \ = \ \int_A K(x,\alpha)\,\mu_1(d\alpha), \qquad A \subseteq \partial_m S_M \tag{3.1}$$

where $\mu_1(d\alpha)$ is the representing measure of $u(x) = 1$; i.e.

$$\int_{\partial_m S_M} K(x,\alpha)\,\mu_1(d\alpha) \ = \ 1, \qquad x \in S \tag{3.2}$$

In particular, since $K(x_0,\alpha) \equiv 1$,

$$P_{x_0}(\lim_{n\to\infty} X_n = X_\infty \in d\alpha) \ = \ \mu_1(d\alpha)$$

and hence $\mu_1(d\alpha)$ is the exit distribution of $\{X_n\}$ for $X_0 = x_0$.

The Dirichlet Problem and the Doob-Naïm-Fatou Theorem

Given a function $g(y)$ on the boundary $\partial\mathcal{O}$ of a domain $\mathcal{O}$, the classical Dirichlet problem is to find a function $u(x)$ that is harmonic in $\mathcal{O}$ and satisfies $\lim_{x\to y} u(x) = g(y)$ for $x \in \mathcal{O}$ and $y \in \partial\mathcal{O}$. If $\partial\mathcal{O}$ is smooth, $u(x) = \int_{\partial\mathcal{O}} g(y)\,\nu_x(dy)$ where $\nu_x(dy)$ is the exit distribution of Brownian motion in $\mathcal{O}$ beginning at x (Doob, 1984). By (1.6), any bounded harmonic function on S is of the form

$$u(x) \ = \ \int_{\partial_m S_M} K(x,\alpha)\,g(\alpha)\,\mu_1(d\alpha)$$

where $g(\alpha)$ is bounded on $\partial_m S_M$. The exit distribution (3.1) suggests that we might have

$$P_x\left(\lim_{n\to\infty} u(X_n) = g(X_\infty)\right) = 1, \qquad \text{all } x \in S$$

It fact, we can conclude more. By (1.3), an arbitrary nonnegative harmonic $u(x) \geq 0$ on S can be written

$$u(x) = \int_{\partial_m S_M} K(x,\alpha)\,\nu(d\alpha), \qquad x \in S \tag{3.3}$$

where $\nu(d\alpha) \geq 0$ on $\partial_m S_M$. One can then show that

$$P_x\left(\lim_{n\to\infty} u(X_n) = q(X_\infty)\right) = 1, \qquad \text{all } x \in S \tag{3.4}$$

where $q(\alpha) = (d\nu/d\mu_1)(\alpha)$ is the absolutely continuous part of the Lebesgue decomposition of $\nu(d\alpha)$ with respect to $\mu_1(d\alpha)$. That is, if $\nu(d\alpha) = g(\alpha)\,\mu_1(d\alpha)$ then $q(\alpha) = g(\alpha)$, and, in general,

$$\nu(d\alpha) = q(\alpha)\,\mu_1(d\alpha) + \nu_c(d\alpha)$$

where $\nu_c(d\alpha)$ and $\mu_1(d\alpha)$ are mutually singular measures on $\partial_m S_M$.

In particular, if $\nu(d\alpha)$ in (3.3) is singular with respect to $\mu_1(d\alpha)$, then

$$P_x\big(\lim_{n\to\infty} u(X_n) = 0\big) = 1 \qquad \text{for all } x \in S$$

The relation (3.4) generalizes a theorem of Fatou about harmonic functions in the upper half-plane.

Using h-Processes to Change the Limit at Infinity

Let $h(x)$ be an nonnegative p-harmonic function on S with $h(x_0) = 1$. Since

$$h(x) = \sum_{y\in S} p(x,y)h(y) = \sum_{y\in S} p_n(x,y)h(y), \qquad x \in S,\ n \geq 1$$

we have $h(x) > 0$ for all $x \in S$ by irreducibility. Set

$$p^h(x,y) = \frac{1}{h(x)}p(x,y)h(y), \qquad x,y \in S \tag{3.5}$$

Then $p^h(x,y) \geq 0$ and $\sum_{y\in S} p^h(x,y) = 1$ for all x, so that (3.5) defines a new transition function $p^h(x,y)$ on S.

We can use $p^h(x,y)$ to define new probability measures $P_x^h(B)$ and a new Markov chain $X_n^{(h)}(w)$ on the Kolmogorov-representation path space (2.3) as before. The resulting Markov process $X_n^{(h)}$ is called the *h-process* of X_n corresponding to $h(x)$. Note that the sample space W and the sample paths $\{X_n^{(h)}(w)\} = \{X_n(w)\}$ are identical in the representation (2.2)–(2.3). Only the probability measures $P_x^h(B)$ and the transition function $p^h(x,y)$ have changed.

The transition functions $p(x,y)$ and $p^h(x,y)$ have essentially the same nonnegative harmonic functions. Specifically, by (3.5), if $u(x) \ge 0$,

$$\sum_{y\in S} p^h(x,y)\frac{u(y)}{h(y)} \;=\; \frac{1}{h(x)}\sum_{y\in S} p(x,y)u(y) \;=\; \frac{u(x)}{h(x)}$$

if and only if $u(x) = \sum_{y\in S} p(x,y)u(y)$. Thus

$$u(x) \text{ is } p\text{-harmonic} \quad \text{if and only} \quad u(x)/h(x) \text{ is } p^h\text{-harmonic} \tag{3.6}$$

It also follows from (3.5) that

$$\begin{aligned} (p^h)_2(x,y) &= \sum_{z\in S} p^h(x,z)\,p^h(z,y) \;=\; \frac{1}{h(x)}\sum_{z\in S} p(x,z)\,p(z,y)h(y) \\ &= \frac{1}{h(x)}p_2(x,y)\,h(y) \end{aligned}$$

and $(p^h)_n(x,y) = \frac{1}{h(x)}p_n(x,y)g(y)$ for all $n \ge 0$ by induction. Similarly, the potential function $g^h(x,y) = \sum_{n=0}^\infty (p^h)_n(x,y) = \frac{1}{h(x)}g(x,y)h(y)$.

The Martin kernel $K(x,y)$ is defined for $x,y \in S$ by

$$K(x,y) \;=\; \frac{g(x,y)}{g(x_0,y)} \qquad \text{for } x,y\in S$$

where $x_0 \in S$ is an arbitrary "reference point" (Section 4). Thus $K(x_0,y) = 1$ for all y. It follows from irreducibility that $K(x,y) \le C_x$ for all $x,y \in S$, where C_x is independent of y (Section 4). The Martin compactification of S is (essentially) the smallest compactification $\widehat{S}$ of S for which the functions $w_x(y) = K(x,y)$ extend in y to continuous functions on $\widehat{S}$, where "smallest" means with the minimal number of points.

The Martin kernel for $p^h(x,y)$ is

$$K^h(x,y) \;=\; \frac{g^h(x,y)}{g^h(x_0,y)} \;=\; \frac{h(x_0)}{h(x)}K(x,y) \;=\; \frac{1}{h(x)}K(x,y) \tag{3.7}$$

Since the continuous functions $w_x(y) = K(x,y)$ defining the Martin compactification are then essentially the same, it follows that $p(x,y)$ and $p^h(x,y)$ have exactly the same Martin boundary (Section 4).

By assumption, $h(x) \ge 0$ is p-harmonic with $h(x_0) = 1$, so that

$$h(x) \;=\; \int_{\partial_m S_M} K(x,\alpha)\mu_h(d\alpha) \tag{3.8}$$

for some $\mu_h(d\alpha) \ge 0$ by (1.3). By (3.7) and (3.8)

$$\int_{\partial_m S_M} K^h(x,\alpha)\mu_h(d\alpha) \;=\; 1, \qquad \text{all } x\in S$$

so that $\mu_h(d\alpha)$ is the representing measure for $u(x) = 1$ for $p^h(x,y)$. Thus

$$P_x^h(\lim_{n\to\infty} X_n = X_\infty \in A) \;=\; \frac{1}{h(x)}\int_A K(x,\alpha)\,\mu_h(d\alpha) \tag{3.9}$$

by (3.1) and (3.7).

For example, the representing measure for the minimal harmonic function $h(x) = h_\beta(x) = K(x,\beta)$ is the Dirac measure $\mu_{h_\beta}(d\alpha) = \delta_\beta(d\alpha)$ for $\beta \in \partial_m S_M$. It then follows from (3.9) that

$$P_x^{h_\beta}(\lim_{n\to\infty} X_n = \beta) \;=\; 1, \qquad \text{all } x \in S$$

Thus, if we want to use probabilistic methods to study the nonnegative harmonic functions $u(x)$, then we can assume that the corresponding Markov chain X_n has any of a wide variety of behaviors as $n \to \infty$.

It turns out that $P_x^{h_\alpha}(B)$ for $\alpha \in \partial_m S_M$ is sufficient to characterize $P_x(B)$ for all events $B \in \mathcal{B}$:

Theorem 3.1. *Let B be an arbitrary event involving the random walk X_n with transition function $p(x,y)$, and let $h_\alpha(x) = K(x,\alpha)$ be the minimal p-harmonic functions for $\alpha \in \partial_m S_M$. Then*

$$P_x(B) \;=\; \int_{\partial_m S_M} P_x^{h_\alpha}(B)\,K(x,\alpha)\,\mu_1(d\alpha) \tag{3.10}$$

where $\mu_1(d\alpha)$ is the representing measure for $u(x) = 1$ in (3.2). More generally,

$$P_x^h(B) \;=\; \frac{1}{h(x)}\int_{\partial_m S_M} P_x^{h_\alpha}(B)\,K(x,\alpha)\,\mu_h(d\alpha) \tag{3.11}$$

for $h(x)$ and $\mu_h(d\alpha)$ in (3.8) with $h(x_0) = 1$.

Proof. It is sufficient to prove (3.10) and (3.11) for cylinder sets

$$B = \{\, w : X_1(w) = y_1,\ X_2(w) = y_2,\ \dots\ X_n(w) = y_n \,\}$$

for $n \ge 0$ and $y_1, y_2, \dots, y_n \in S$. The general case follows by extension theorems for measures. By (2.2) and (3.5),

$$\begin{aligned} P_x^h(B) \;&=\; p^h(x,y_1)p^h(y_1,y_2)\dots p^h(y_{n-1},y_n) \\ &=\; \frac{1}{h(x)}p(x,y_1)p(y_1,y_2)\dots p(y_{n-1},y_n)h(y_n) \\ &=\; \frac{1}{h(x)}E_x\big(I_B h(X_n)\big) \end{aligned}$$

In particular, $P_x^{h_\alpha}(B) = \frac{1}{K(x,\alpha)}E_x\big(I_B K(X_n,\alpha)\big)$. Thus by (3.8)

$$\begin{aligned} P_x^h(B) \;&=\; \frac{1}{h(x)}\int_{\partial_m S_M} E_x\big(I_B K(X_n,\alpha)\big)\,\mu_h(d\alpha) \\ &=\; \frac{1}{h(x)}\int_{\partial_m S_M} P_x^{h_\alpha}(B)\,K(x,\alpha)\,\mu_h(d\alpha) \end{aligned}$$

Setting $h(x) = 1$ implies (3.10).

Note. If $p(x,y)$ is irreducible and defines a *random walk* on S — i.e., $p(gx, gy) = p(x,y)$ for all $x, y \in S$ and $g \in G$ — then $p^h(x,y) = \frac{1}{h(x)} p(x,y)h(y)$ defines a random walk if and only if $h(x)$ is *multiplicative*; i.e.

$$\frac{h(gx)}{h(x)} = \frac{h(gy)}{h(y)} = \phi(g) \qquad \text{for all } x, y \in S, \ \ g \in G \tag{3.12}$$

for some function $\phi(g)$ on G. If $S = G$, (3.12) implies $h(xy) = h(x)h(y)$.

If S is an Abelian group, multiplicative harmonic functions are the same as minimal harmonic functions, but not necessarily if S is not an Abelian group.

4. Construction and Representation

In this section we construct the Martin boundary and prove the basic representation theorem (1.2) for non-negative p-harmonic functions. As in Sections 1–3, we assume that S is a discrete, countable set and assume that $p(x,y)$ satisfies:

$$p(x,y) \geq 0, \qquad \sum_{y \in S} p(x,y) = 1 \qquad \text{for all } x \in S$$

Let $p_n(x,y)$ be the n^{th} matrix power of $p(x,y)$ as before, and assume

$$g(x,y) \ = \ \sum_{n=0}^{\infty} p_n(x,y) \ < \ \infty, \qquad \text{all } x, y \in S$$

Irreducibility implies that $g(x_0, y) > 0$ for all $x_0, y \in S$. Here we only assume that $g(x_0, y) > 0$ for all $y \in S$ and some fixed x_0. This is weaker than irreducibility, and is equivalent to assuming that you can get to any $y \in S$ from some fixed point $x_0 \in S$. This will be important in Section 6. Define

$$K(x,y) \ = \ \frac{g(x,y)}{g(x_0,y)}, \qquad x, y \in S \tag{4.1}$$

Then

Lemma 4.1. *If $g(x_0, y) > 0$ for all $y \in S$, then there exist constants C_x, independent of y, such that*

$$K(x,y) \ = \ \frac{g(x,y)}{g(x_0,y)} \ \leq \ C_x \qquad \textit{for all } x, y \in S$$

Proof. Given $x \in S$, $p_m(x_0, x) > 0$ for some $m \geq 0$ since $g(x_0, x) > 0$. In general

$$\sum_{z \in S} p_m(x,z)\, g(z,y) \ = \ \sum_{z \in S} \sum_{n=0}^{\infty} p_m(x,z) p_n(z,y) \ = \ \sum_{n=0}^{\infty} p_{m+n}(x,y) \tag{4.2}$$
$$= \ \sum_{n=m}^{\infty} p_n(x,y) \ = \ g(x,y) - \sum_{n=0}^{m-1} p_n(x,y)$$

Thus

$$g(x_0, y) \;\geq\; \sum_{z\in S} p_m(x_0, z)\, g(z, y) \;\geq\; p_m(x_0, x) g(x, y)$$

and

$$K(x, y) \;=\; \frac{g(x, y)}{g(x_0, y)} \;\leq\; C_x = \frac{1}{p_m(x_0, x)}$$

The next step in the construction of $\widehat{S}_M$ is to define a metric $\rho(x, y)$ on S such that the completion of the metric space (S, ρ) is a compact space $\widehat{S}$ with the desired properties. Set

$$\rho(x, y) \;=\; \sum_{q\in S} D(q)\, \frac{|K(q, x) - K(q, y)| + |\delta_{qx} - \delta_{qy}|}{C_q + 1} \tag{4.3}$$

where δ_{xy} is the Kronecker delta, the constants C_q are the same as in Lemma 4.1, and $D(q)$ satisfy $\sum_{q\in S} D(q) < \infty$. Then the series (4.3) converges uniformly in x and y. Also, $\rho(y, y) = 0$, $\rho(y, z) > 0$ if $y \neq z$, $\rho(y, w) \leq \rho(y, z) + \rho(z, w)$ for $y, z, w \in S$, and $\rho(y, z)$ is a metric on S.

A sequence $\{\, y_n \,\} \subseteq S$ is a Cauchy sequence for $\rho(y, z)$ (i.e., $\lim_{m,n\to\infty} \rho(y_n, y_m) = 0$) if and only if EITHER (i) $y_n \equiv y$ for all $n \geq n_0$ for some $n_0 < \infty$ and $y \in S$ (so that $\{\, y_n \,\}$ converges in S), OR ELSE (ii) $\lim_{n\to\infty} y_n = \infty$ and the limits $\lim_{n\to\infty} K(x, y_n)$ exist for every $x \in S$. (Here $\lim_{n\to\infty} y_n = \infty$ means that y_n eventually leaves every finite set and never returns.)

An arbitrary metric space (S, ρ) can be embedded in a complete metric space $(\widehat{S}, \rho)$ by considering equivalence classes of Cauchy sequences. In this case, since the series (4.3) converges uniformly, any sequence $\{\, y_n \,\} \subseteq S$ has a subsequence that is a Cauchy sequence by diagonalization. This implies that the complete metric space $(\widehat{S}, \rho)$ is compact.

Since S is discrete, $K(x, y)$ is a continuous function of $y \in S$ for each fixed $x \in S$. By (4.3),

$$|K(x, y) - K(x, z)| \;\leq\; \frac{C_x + 1}{D(x)}\, \rho(y, z)$$

Thus, for each $x \in S$, $K(x, y)$ has a unique extension to $\widehat{S}$ as a continuous function of y. Since $K(x_0, y) = 1$ for $y \in S$,

$$K(x_0, z) = 1 \qquad \text{for all } z \in \widehat{S}$$

The space $(\widehat{S}, \rho)$ for $\rho(y, z)$ in (4.3) is the Martin compactification $\widehat{S}_M$ of S, and $\partial S_M = \widehat{S}_M - S$ is the Martin boundary. Note that both depend on the transition function $p(x, y)$ and perhaps on the reference point $x_0 \in S$ as well.

The terms δ_{xy}, δ_{xz} in the definition (4.3) of $\rho(y, z)$ are *not* required to make $\rho(y, z)$ a metric. If $\rho(y, z) = 0$ *without* these terms in (4.3), then $K(x, y) = K(x, z)$ for all $x \in S$. Then $p_x K(x, y) = p_x K(x, z)$ for all x, where $pf(x) = \sum_{y\in S} p(x, y) f(y)$, and $\delta_{xy}/g(x_0, y) = \delta_{xz}/g(x_0, z)$ by (4.2). Thus $y = z$ and $\rho(y, z)$ is a metric *without* the terms δ_{xy}, δ_{xz} in (4.3).

The reason for using these terms is more subtle. Suppose that we had a sequence $\{\, y_n \,\} \subseteq S$ and $y_0 \in S$ such that

$$\lim_{n\to\infty} K(x, y_n) \;=\; K(x, y_0) \quad \text{with } y_n \to \infty,\ y_0 \in S, \quad \text{all } x \in S \tag{4.4}$$

Then $\lim_{n\to\infty} \rho(y_n, y_0) = 0$ if $\rho(y, z)$ were defined in (4.3) *without* the terms δ_{xy}, δ_{xz}, and what should be a non-convergent Cauchy sequence $\{ y_n \} \subseteq S$ converges to $y_0 \in S$. The result is that the set S would not be *open* in the completion $\widehat{S}$, and the boundary $\widehat{S} - S$ would not be a *closed* set in $\widehat{S}$. The terms δ_{xy}, δ_{xz} in (4.3) prevent this, since then $\rho(y_0, y_n) \geq D(y_0)/(C_{y_0} + 1) > 0$ whenever $y_n \neq y_0$.

Thus these terms in (4.3) guarantee that S is an open set in the Martin compactification $\widehat{S}_M$, and that the Martin boundary $\partial S_M = \widehat{S}_M - S$ is closed. By (4.2), $K(x, \alpha) = K(x, y_0)$ is not p-harmonic in x in either case, so that $\alpha \notin \partial_m S_M$ and the minimal Martin boundary representation for p-harmonic functions is unaffected.

Whether the terms δ_{xy}, δ_{xz} (or their equivalent) are included in $\rho(y, z)$ or not is a matter of taste. One can usually prove that (4.4) does not happen in non-pathological cases, but it still has to be checked if you want to assert that ∂S_M is closed with the simpler definition of $\rho(y, z)$. Doob's (1959) original construction of the Martin boundary did not automatically guarantee that S is open in $\widehat{S}_M$. Along with other authors, we have chosen to define the Martin boundary in such a way that the boundary ∂S_M is always closed even if (4.4) can occur.

Before proving the Martin representation formula (1.2) for p-harmonic functions, we need to introduce p-superharmonic functions.

Superharmonic Functions and the Riesz Representation Theorem

Set $pu(x) = \sum_{y\in S} p(x, y)u(y)$ for functions $u(x) \geq 0$ on S. Thus $u(x) \geq 0$ is p-harmonic if and only if $u(x) = pu(x)$ for all x. Similarly, we say that $u(x) \geq 0$ is *p-superharmonic* if $0 \leq pu(x) \leq u(x)$; i.e.

$$u(x) \ \geq \ pu(x) \ = \ \sum_{y\in S} p(x, y)\, u(y), \qquad \text{all } x \in S \tag{4.5}$$

A function $u(x)$ is called a *potential* on S if we can write

$$u(x) \ = \ \sum_{y\in S} g(x, y)\, k(y) \qquad \text{for some function } k(x) \geq 0$$

Since then $pu(x) = \sum_{y\in S} p_x g(x, y)k(y) = u(x) - k(x) \leq u(x)$ by (4.2), every potential is automatically superharmonic. A theorem of Riesz in classical potential theory says that every superharmonic function can be written uniquely as the sum of a potential and a harmonic function. The proof is easier in our case, but the result is no less important.

Lemma 4.2 (Riesz Representation Theorem). *Every p-superharmonic function $u(x) \geq 0$ can be written*

$$u(x) \ = \ \sum_{y\in S} g(x, y)\, k(y) \ + \ h(x) \tag{4.6}$$

where $k(y) \geq 0$ and $h(x)$ is p-harmonic. The representation (4.6) is unique, with $k(x) = u(x) - pu(x)$ and $h(x) = \lim_{n\to\infty} p_n u(x)$.

Proof. Since $u(x) \geq pu(x) \geq 0$, it follows that $p_n u(x) \geq p_{n+1} u(x) \geq 0$ for all n and $p_n u(x) \downarrow$ in n for all x. Thus $h(x) = \lim_{n\to\infty} p_n u(x)$ always exists, and is

p-harmonic by the same argument. In general

$$\begin{aligned} u(x) &= \sum_{n=0}^{m-1} \big(p_n u(x) - p_{n+1} u(x)\big) \;+\; p_m u(x) \\ &= \sum_{n=0}^{m-1} p_n(x,y)\big(u(y) - pu(y)\big) \;+\; p_m u(x) \end{aligned}$$

Since $u(x) - pu(x) \ge 0$, (4.6) follows by letting $m \to \infty$. Conversely, if (4.6) holds, then $pu(x) = u(x) - k(x)$ by (4.2).

Similar arguments prove the following result:

Lemma 4.3. (i) *If $u(x) \ge 0$ is p-superharmonic, then $u(x)$ is a potential if and only if $\lim_{n\to\infty} p_n u(x) = 0$ for all $x \in S$.*

(ii) *If $0 \le u(x) \le v(x)$ for all x, where $u(x)$ is p-superharmonic and $v(x)$ is a potential, then $u(x)$ is also a potential.*

(iii) *If $u(x) \ge 0, v(x) \ge 0$ are p-superharmonic, then $w(x) = u(x) \wedge v(x) = \min\{u(x), v(x)\}$ is also p-superharmonic.*

(iv) *$K(x, \alpha)$ is p-superharmonic in x for each $\alpha \in \partial S_M$.*

Proofs. (i) This follows directly from Lemma 4.2, since then $h(x) = 0$.

(ii) Since $0 \le p_n u(x) \le p_n v(x)$, this follows from part (i).

(iii) Thus $pw(x) \le pu(x) \le u(x)$ and $pw(x) \le pv(x) \le v(x)$, from which $pw(x) \le u(x) \wedge v(x) = w(x)$ follows.

(iv) Since $K(x,y) = g(x,y)/g(x_0,y)$ for $x, y \in S$, $K(x,y)$ is a potential (and is hence p-superharmonic) as a function of x for each $y \in S$. If $y_n \to \alpha \in \partial S_M$, then by Fatou's Lemma

$$\sum_{z\in S} p(x,z) \lim_{n\to\infty} K(z, y_n) \;\le\; \lim_{n\to\infty} \sum_{z\in S} p(x,z) K(z, y_n) \;\le\; \lim_{n\to\infty} K(x, y_n)$$

and hence $K(x, \alpha)$ is p-superharmonic as a function of x.

Proof of the Martin Representation Theorem (1.2)

Theorem 4.1. *Let $p(x,y)$ be a transition function on a discrete countable set S. Assume (i) $g(x,y) < \infty$ for all $x, y \in S$ and (ii) $g(x_0, y) > 0$ for all $y \in S$ for some $x_0 \in S$. Define the Martin kernel $K(x,y)$ and Martin boundary ∂S_M as in (4.3)-(4.4). Then, for any p-harmonic function $u(x) \ge 0$, there exists a measure $\mu(d\alpha) \ge 0$ on ∂S_M such that*

$$u(x) \;=\; \int_{\partial S_M} K(x, \alpha)\, \mu(d\alpha), \qquad x \in S \tag{4.7}$$

Proof. We use the classical "method of balayage." Since S is countable, we can choose finite sets $S_n \uparrow S$. Define

$$\begin{aligned} u_n(x) &= u(x) \wedge \Big\{ n \sum_{y \in S_n} g(x,y) \Big\} \\ &= u(x) \wedge w_n(x), \qquad w_n(x) = \sum_{y\in S} g(x,y) n I_{S_n}(y) \end{aligned}$$

Since $u(x)$ is p-harmonic and $w_n(x)$ is a potential, each function $u_n(x)$ is a potential by Lemma 4.3(iii,ii). Thus

$$u_n(x) \;=\; \sum_{y\in S} g(x,y)\,k_n(y) \;=\; \sum_{y\in S} \frac{g(x,y)}{g(x_0,y)}\,g(x_0,y)k_n(y) \tag{4.8}$$
$$= \int_S K(x,y)\,\mu_n(dy), \qquad \mu_n(\{y\}) = g(x_0,y)k_n(y)$$

where $k_n(y) \geq 0$. Since $K(x_0,y) = 1$, $\mu_n(S) = u_n(x_0) \leq u(x_0)$. Thus $\mu_n(dy)$ are a bounded sequence of measures on the compact metric space $\widehat{S}_M$, so that a subsequence $\lim_{k\to\infty}\mu_{n_k}(dy) = \mu(dy)$ converges weakly over continuous functions on $\widehat{S}_M$. Since $K(x,y)$ is continuous in $y \in \widehat{S}_M$ for each fixed x, the right-hand side of (4.8) for $n = n_k$ converges to $\int_{\widehat{S}_M} K(x,y)\,\mu(dy)$. Since $w_n(x) \geq ng(x,x) \geq n$ for $x \in S_n$, the functions $w_n(x) \uparrow \infty$ and $0 \leq u_n(x) \uparrow u(x)$ for all x. Hence, by (4.8)

$$u(x) \;=\; \int_{\widehat{S}_M} K(x,y)\,\mu(dy)$$
$$= \sum_{y\in S} g(x,y)\frac{\mu(\{y\})}{g(x_0,y)} \;+\; \int_{\partial S_M} K(x,\alpha)\,\mu(d\alpha)$$

Finally, by (4.2),

$$0 \;=\; u(x) - pu(x) \;=\; \frac{\mu(\{x\})}{g(x_0,x)} \;+\; \int_{\partial S_M} \big(K(x,\alpha) - p_x K(x,\alpha)\big)\,\mu(d\alpha)$$

By Lemma 4.1(iv), this implies both that $\mu(S) = 0$ (so that (4.7) holds) and that $\mu(d\alpha)$ is concentrated on the set of $\alpha \in \partial S_M$ for which $K(x,\alpha)$ is p-harmonic in x.

Corollary 4.1. *Let $u(x) \geq 0$ be any p-superharmonic function on S. Then, under the same assumptions as Theorem 4.1, there exists a measure $\mu(d\alpha) \geq 0$ on $\widehat{S}_M$ such that*

$$u(x) \;=\; \int_{\widehat{S}_M} K(x,\alpha)\,\mu(d\alpha), \qquad x \in S$$

Proof. Combine Lemma 4.2 and Theorem 4.1.

5. Limits at the Boundary

Assuming the minimal Martin representation (1.3), the purpose of this section is to prove

Theorem 5.1. *Under the assumptions of Sections 1 and 2*

$$P_x(\lim_{n\to\infty} X_n = X_\infty \in A) \;=\; \int_A K(x,\alpha)\,\mu_1(d\alpha) \tag{5.1}$$

for all Borel subsets $A \subseteq \partial_m S_M$, where $\mu_1(d\alpha)$ is the representing measure for $u(x) = 1$:

$$\int_{\partial_m S_M} K(x,\alpha)\,\mu_1(d\alpha) \;=\; 1, \qquad \text{all } x \in S$$

The proof will be by a series of reductions. The reader should be warned that the proof of (5.1) is a bit subtle in places.

First, define

$$\tau_F(w) \;=\; \min\{\, n \geq 0 : X_n(w) \in F\,\}$$

for an arbitrary subset $F \subseteq S$, with $\tau_F(w) = \infty$ if $X_n(w) \notin F$ for all $n \geq 0$. Then

Lemma 5.1. *Let $F \subseteq S$ be a finite set. Then there exists a function $k_F(x) \geq 0$ with support in F such that*

$$P_x(\tau_F < \infty) \;=\; \sum_{y\in F} g(x,y)k_F(y) \;=\; \sum_{y\in S} g(x,y)I_F(y)k_F(y)$$

where $I_F(x)$ is the indicator function of the set F.

Proof. Let $u_F(x) = P_x(\tau_F < \infty)$. By arguing from (2.2)

$$\sum_{y\in S} p(x,y)u_F(y) \;=\; P_x(X_k \in F \text{ for some } k \geq 1)$$

and for $n \geq 1$

$$\sum_{y\in S} p_n(x,y)u_F(y) \;=\; P_x(X_k \in F \text{ for some } k \geq n) \tag{5.2}$$

In particular

$$u_F(x) - pu_F(x) \;=\; k_F(x) \;=\; P_x(X_0 \in F,\ X_k \notin F \text{ for } k \geq 1) \;\geq\; 0 \tag{5.3}$$

Hence $u_F(x)$ is superharmonic. Since F is finite, $\lim_{n\to\infty} p_n u_F(x) = 0$ by (5.2), and $u_F(x)$ is a potential with $k(x) = k_F(x) = I_F(x)k_F(x)$ by the Riesz Representation Theorem (Lemma 4.2).

It follows similarly from (5.2)–(5.3) and Lemma 4.2 that

Corollary 5.1. *For an arbitrary subset $A \subseteq S$,*

$$v_A(x) = P_x(X_n \in A \;\text{ for infinitely many } n)$$

is a p-harmonic function of x.

The next step is

Lemma 5.2. *For an arbitary subset $A \subseteq S$, there exists a measure $\nu_A(d\alpha) \geq 0$ on ∂S_M such that*

$$v_A(x) = P_x(X_n \in A \;\text{ for infinitely many } n) \;=\; \int_{\overline{A}\cap\partial S_M} K_0(x,\alpha)\,\nu_A(d\alpha) \tag{5.4}$$

where $K_0(x,\alpha) = \lim_{n\to\infty} p_n K(x,\alpha)$ is the harmonic part of the superharmonic function $K(x,\alpha)$.

Remark. It follows from Corollary 5.1 that $v_A(x)$ is harmonic, and hence $v_A(x) = \int_{\partial S_M} K(x,\alpha)\,\mu_A(d\alpha) = \int_{\partial S_M} K_0(x,\alpha)\,\mu_A(d\alpha)$ by Theorem 4.1 and Lemma 4.3(iv). The crucial point of Lemma 5.2 is that $\mathrm{Supp}(\nu_A) \subseteq \overline{A}$.

Proof. Since X_n is transient, Lemma 5.2 holds with $\nu_A = 0$ if A is finite. Choose finite sets $A_m \uparrow A$ as $m \to \infty$. Then by Lemma 5.1

$$P_x(\tau_{A_m} < \infty) \;=\; \int_{A_m} K(x,y)\,\nu_m(dy), \qquad m \geq 1$$

where $\mathrm{Supp}(\nu_m) \subseteq A_m \subseteq A$. Since $P_{x_0}(\tau_{A_m} < \infty) = \nu_m(S) \leq 1$, there exists a weakly convergent subsequence $\nu_{m_k}(dy) \to \nu_A(dy)$ in $\widehat{S}_M$. Since the random variables $\tau_{A_m} \downarrow \tau_A$ as $m \to \infty$,

$$u_A(x) \;=\; P_x(\tau_A < \infty) \;=\; \int_{\overline{A}} K(x,y)\,\nu_A(dy) \tag{5.5}$$

which generalizes Lemma 5.1 to arbitrary sets $A \subseteq S$. By (5.2), as $n \to \infty$,

$$\begin{aligned} p_n u_A(x) = P_x(X_k \in A,\ \text{some } k \geq n) \;&=\; \int_{\overline{A}} p_n K(x,y)\,\nu_A(dy) \\ &\to P_x(X_k \in A \text{ for infinitely many } k) \qquad \text{for all } x \in S \end{aligned} \tag{5.6}$$

Since $K(x,y)$ is superharmonic in x, $p_n K(x,y) \downarrow K_0(x,y)$ for $y \in \widehat{S}$, and $(p_n)_x K(x,y) \downarrow 0$ for $y \in S$ by (4.1) and (4.2). The relation (5.4) then follows from (5.6).

Lemma 5.3 (Zero-One Law). *Suppose that $u_1(x) = 1$ is minimal harmonic (or, equivalently, that all bounded harmonic functions are constant). Then, for any subset $A \subseteq S$,*

EITHER $\quad P_x(X_k \in A$ *for infinitely many* $k) = 0, \qquad$ *all* $x \in S$,

OR ELSE $\quad P_x(X_k \in A$ *for infinitely many* $k) = 1, \qquad$ *all* $x \in S$

Remark. This follows from the Kolmogorov Zero-One Law for the special case of random walks in R^d. Lemma 5.3 extends Kolmogorov's proof to any irreducible transient Markov chain for which all bounded harmonic functions are constant. See Section 7 for a discussion of random walks on Z^d.

Proof. Let $u(x) = P_x(X_k \in A$ for infinitely many $k)$. Since $u(x)$ is harmonic by Lemma 5.2, $u(x) = c$ for all $x \in S$ for some constant c. We will prove that $c^2 = c$, from which it follows that either $c = 0$ or $c = 1$.

It is sufficient to assume that the underlying probability space $(W, \mathcal{B})$ is the Kolmogorov Representation space

$$W \;=\; S^\infty \;=\; \{\, w = (a_0, a_1, \ldots, a_n, \ldots) : \;\; a_i \in S \,\}$$

with $X_n(w) = a_n$ (see (2.3)). The σ-algebra $\mathcal{B} = \mathcal{B}(W)$ is generated by the cylinder sets

$$C = C_{n;\, b_0,\ldots,b_n} = \{\, w = (a_0, a_1, \ldots, a_m, \ldots) : a_i = b_i \;\; \text{for } 0 \leq i \leq n \,\} \tag{5.7}$$

Set $E = \{\, w : X_n(w) \in A$ for infinitely many $n \,\}$. Then, for each $x \in S$ and $\epsilon > 0$, there exists a finite disjoint union of cylinder sets $F = \cup_{i=1}^n C_i$ such that $P_x(F \Delta E) =$

$P_x(F - E) + P_x(E - F) < \epsilon$. In particular, $|P_x(F) - P_x(E)| = |P_x(F) - c| < \epsilon$. For any cylinder set C,

$$\begin{aligned} P_x(C \cap E) &= P_x(X_0 = b_0, \dots, X_n = b_n,\ X_k \in A \text{ for infinitely many } k \geq n) \\ &= P_x(C)\, P_{b_n}(E) \ = \ P_x(C)\, c \end{aligned}$$

by (2.2). Hence

$$P_x(F \cap E) = \sum_{i=1}^{n} P_x(C_i \cap E) = \sum_{i=1}^{n} P_x(C_i)\, c = P_x(F)\, c \ \leq \ (c+\epsilon)c \ \leq \ c^2 + \epsilon$$

Similarly $P_x(F \cap E) = P_x(E) - P_x(E - F) > c - \epsilon$. Thus $c^2 \leq c \leq c^2 + 2\epsilon$ and $c = c^2$, which implies that either $c = 0$ or $c = 1$.

Proof of Theorem 5.1. By Theorem 3.1, for any Borel set $A \subseteq \partial_m S_M$,

(5.8)
$$P_x(\lim_{n\to\infty} X_n = X_\infty \in A) \ = \ \int_{\partial_m S_M} P_x^{h_\alpha}(\lim_{n\to\infty} X_n = X_\infty \in A)\, K(x,\alpha)\, \mu_1(d\alpha)$$

where $h_\alpha(x) = K(x,\alpha)$. If we can prove

$$P_x^{h_\alpha}(\lim_{n\to\infty} X_n = \alpha) \ = \ 1 \qquad \text{for all } x \in S \text{ and } \alpha \in \partial_m S_M \tag{5.9}$$

then the right-hand side of (5.8) will reduce to $\int_{A\cap\partial_m S_M} K(x,\alpha)\, \mu_1(d\alpha)$, which is Theorem 5.1.

Fix $\alpha \in \partial_m S_M$ in (5.9) and set $A_\epsilon = \{\, x \in S : \rho(x,\alpha) \geq \epsilon \,\}$. In particular, $\alpha \notin \overline{A_\epsilon}$. If we can show

$$u(x) \ = \ P_x^{h_\alpha}(X_n \in A_\epsilon \text{ for infinitely many } n) \ = \ 0 \tag{5.10}$$

for all $x \in S$ and $\epsilon > 0$, then (5.9) will follow and hence Theorem 5.1.

Note that $u(x)$ is p^{h_α}-harmonic by Corollary 5.1. In general, by (3.6), a function $w(x)$ is p^{h_α}-harmonic if and only if $w(x)K(x,\alpha)$ is p-harmonic. Since $K(x,\alpha)$ is minimal p-harmonic, all bounded p^{h_α}-harmonic functions $w(x)$ are constant. Thus, by Lemma 5.3, either $u(x) = 0$ in (5.10) for all $x \in S$ or $u(x) = 1$ for all $x \in S$, and we only have to exclude the case $u(x) \equiv 1$.

By Lemma 5.2 and (3.7), and the relation $p_n^h(x,y) = \frac{1}{h(x)} p_n(x,y) h(y)$,

$$u(x) \ = \ \int_{\overline{A_\epsilon}\cap\partial S_M} K_0^{h_\alpha}(x,\beta)\, \nu_\epsilon(d\beta) \ = \ \int_{\overline{A_\epsilon}\cap\partial S_M} \frac{K_0(x,\beta)}{K(x,\alpha)}\, \nu_\epsilon(d\beta) \tag{5.11}$$

for some measure $\nu_\epsilon(d\beta) \geq 0$. Since $u(x) \leq 1$, $\int_A K_0(x,\beta)\, \nu_\epsilon(d\beta) \leq K(x,\alpha)$ for all $A \subseteq \overline{A_\epsilon} \cap \partial S_M$. Since $K(x,\alpha)$ is minimal harmonic,

$$\int_A K_0(x,\beta)\, \nu_\epsilon(d\beta) \ = \ C_A\ K(x,\alpha) \ = \ \int_A K_0(x_0,\beta)\, \nu_\epsilon(d\beta)\ \ K(x,\alpha)$$

for arbitrary $A \subseteq \overline{A_\epsilon} \cap \partial S_M$, where we evaluated the constant C_A by setting $x = x_0$. Thus

$$K_0(x,\beta) = K_0(x_0,\beta)\, K(x,\alpha) \qquad \text{for } \nu_\epsilon(d\beta) \text{ a.e. } \beta \in \overline{A_\epsilon} \cap \partial S_M \tag{5.12}$$

If (5.10) is false, then $u(x_0) = \int_{\overline{A_\epsilon}\cap\partial S_M} K_0(x_0,\beta)\,\nu_\epsilon(d\beta) = 1$ by (5.11) and Lemma 5.3. Since $\nu_\epsilon(\overline{A_\epsilon}) = P_{x_0}(\tau_{A_\epsilon} < \infty) \le 1$ by (5.5), and since $K_0(x_0,\beta) \le K(x_0,\beta) = 1$, it follows from (5.11) that $K_0(x_0,\beta) = 1$ for $\nu_\epsilon(d\beta)$-almost every $\beta \in \overline{A_\epsilon}\cap\partial S_M$. Thus $K_0(x,\beta) = K(x,\alpha)$ by (5.12) for $\nu_\epsilon(d\beta)$-a.e. $\beta \in \overline{A_\epsilon}\cap\partial S_M$ and all $x \in S$.

By Lemma 4.2, $K(x,\beta) = K_0(x,\beta) + g_\beta(x)$ where $g_\beta(x) = \sum_{y\in S} g(x,y)k_\beta(y)$ for $k_\beta(y) \ge 0$. If $K(x_0,\beta) = K_0(x_0,\beta) = 1$, then $g_\beta(x_0) = 0$ and thus $g_\beta(x) = 0$ for all $x \in S$. Thus $K_0(x,\beta) = K(x,\beta) = K(x,\alpha)$ for all $x \in S$ and $\nu_\epsilon(d\beta)$-a.e. $\beta \in \overline{A_\epsilon}\cap\partial S_M$.

It follows from the construction of the Martin boundary in Section 4 that if $K(x,\alpha) = K(x,\beta)$ for all $x \in S$ for $\alpha,\beta \in \partial S_M$, then $\alpha = \beta$. Since $\alpha \notin \overline{A_\epsilon}$, it follows that $\nu_\epsilon(\overline{A_\epsilon}\cap\partial S_M) = 0$. Thus $u(x) = 0$ in (5.11) and (5.10) for all $\epsilon > 0$, which completes the proof of (5.1) and hence of Theorem 5.1.

6. Competition Games and an Urn Model

Let $\{ Z_n \}$ be the random walk in Z^2 with transition function

$$p[\,(x,y),\ (x+1,y)\,] \;=\; p[\,(x,y),\ (x,y+1)\,] \;=\; \frac{1}{2} \tag{6.1}$$

That is, given $Z_n = z \in Z^2$, the process either goes up by one step or else goes to the right by one step, with probability $1/2$ in both cases. Although this is one of the simplest random walks in Z^2, it has an interesting Martin boundary theory. The Martin representation theorem of Section 4 will provide a solution for a special case of the classical Hausdorff moment problem. An h-process of (6.1) will lead to a nontrivial asymptotic result for an urn model (specifically, a strong law of large numbers with a random limit).

Markov processes of the form (6.1) can be called competition games, since if $Z_n = (X_n, Y_n)$

$$|Z_n| \;=\; X_n + Y_n \;=\; X_0 + Y_0 + n \tag{6.2}$$

and one can ask how the n changes between times 0 and n are distributed between $X_n - X_0$ and $Y_n - Y_0$.

While the process $\{ Z_n \}$ is not irreducible, we do have that $g(z_0, w) > 0$ for $z_0 = (0,0)$ and $w = (x,y)$ for $x \ge 0, y \ge 0$. That is, if

$$S \;=\; Q_{(0,0)} \;=\; \{\, (x,y) \in Z^2 :\ \ x \ge 0,\ y \ge 0 \,\}$$

then $g(z_0, w) > 0$ for all $w \in S$. This is sufficient for the results of Section 4.

By (6.1), the infinite series $g(z,w) = \sum_{n=0}^{\infty} p_n(z,w)$ has at most one positive term. If $m \ge x$ and $n \ge y$,

$$\begin{aligned} g[\,(x,y),\ (m,n)\,] \;&=\; p_{m+n-x-y}[\,(x,y),\ (m,n)\,] \\ &=\; \binom{m-x+n-y}{m-x}\left(\frac{1}{2}\right)^{m+n-x-y} \end{aligned}$$

$$g[\,(0,0),\ (m,n)\,] \;=\; p_{m+n}[\,(x,y),\ (m,n)\,] \;=\; \binom{m+n}{m}\left(\frac{1}{2}\right)^{m+n}$$

with $g[\,(x,y),\ (m,n)\,] = 0$ if $m < x$ or $n < y$. Hence

$$\begin{aligned} K[\,(x,y),\ (m,n)\,] &= \frac{g[\,(x,y),\ (m,n)\,]}{g[\,(0,0),\ (m,n)\,]} = \frac{\binom{m-x+n-y}{m-x}\left(\frac{1}{2}\right)^{m+n-x-y}}{\binom{m+n}{m}\left(\frac{1}{2}\right)^{m+n}} \\ &= 2^{x+y}\frac{m(m-1)\dots(m-x+1)\,n(n-1)\dots(n-y+1)}{(m+n)(m+n-1)\dots(m+n-x-y+1)} \\ &= 2^{x+y}\left(\frac{m}{m+n}\right)^x\left(\frac{n}{m+n}\right)^y\left(1+O\Big(\frac{x^2}{m}\Big)+O\Big(\frac{y^2}{n}\Big)\right) \end{aligned}$$

Thus

Theorem 6.1. *Assume $(m_k,n_k)\to\infty$ (i.e., $\max\{m_k,n_k\}\to\infty$). Then*

$$\lim_{k\to\infty} K[\,(x,y),\ (m_k,n_k)\,] \quad \text{exists}, \qquad \text{all } (x,y)\in S \tag{6.3}$$

if and only if

$$\lim_{k\to\infty}\frac{m_k}{m_k+n_k} = \alpha \quad \textit{exists for some } \alpha, \quad 0\le\alpha\le 1$$

In that case, the limit in (6.3) equals

$$K[\,(x,y),\ \alpha\,] = 2^{x+y}\,\alpha^x(1-\alpha)^y \tag{6.4}$$

with the convention $0^0=1$ and $0^x=0$ for $x>0$.

Corollary 6.1. *A function $h(x,y)\ge 0$ is p-harmonic on S — that is,*

$$h(x,y) = \frac{h(x+1,y)\ +\ h(x,y+1)}{2}, \qquad \textit{all } (x,y)\in S \tag{6.5}$$

if and only if there exists a measure $\mu(d\alpha)\ge 0$ on $I=[0,1]$ such that

$$h(x,y) = 2^{x+y}\int_0^1 \alpha^x(1-\alpha)^y\,\mu(d\alpha), \qquad \textit{all } (x,y)\in S \tag{6.6}$$

It turns out that the Corollary is essentially equivalent to a special case of the classical Hausdorff moment problem (Widder, 1946); see below.

The functions $h(x,y) = K[\,(x,y),\ \alpha\,]$ in (6.4) are all harmonic; i.e., satisfy (6.5). Moreover

Lemma 6.1. *The function $K[\,(x,y),\ \alpha\,]$ is minimal harmonic for $0\le\alpha\le 1$. That is, if*

$$h(x,y) = \int_0^1 K[\,(x,y),\ s\,]\,\mu(ds) \ \le\ CK[\,(x,y),\ \alpha\,] \qquad \textit{all } (x,y)\in S \tag{6.7}$$

then $h(x,y) = cK[\,(x,y),\ \alpha\,]$ for some constant $c\le C$.

Proof. If $0<\alpha<1$ in (6.7), then by (6.4) and (6.7)

$$\int_0^1\left(\frac{s}{\alpha}\right)^x\left(\frac{1-s}{1-\alpha}\right)^y\mu(ds) \ \le\ C, \qquad \text{all } x\ge 0,\ y\ge 0$$

Fixing y and letting $x\to\infty$ implies $\mathrm{Supp}(\mu)\subseteq[0,\alpha]$. Fixing x and letting $y\to\infty$ implies $1-s\le 1-\alpha$ on $\mathrm{Supp}(\mu)$, or $\mathrm{Supp}(\mu)\subseteq[\alpha,1]$. Thus $\mathrm{Supp}(\mu)=\{\,\alpha\,\}$ and $h(x,y)=cK[\,(x,y),\ \alpha\,]$ for $c=\mu(\{\alpha\})$.

If $\alpha=0$, then $K[\,(x,y),\ \alpha\,]=0$ if $x>0$, which implies $\mathrm{Supp}(\mu)=\{\,0\,\}$ by (6.7). Similarly, $K[\,(x,y),\ 1\,]=0$ if $y>0$, which implies $\mathrm{Supp}(\mu)=\{\,1\,\}$.

Corollary 6.2. *If $h(x,y)$ is harmonic and bounded for $p(z,w)$ in (6.1), then $h(x,y)$ is constant.*

Proof. Note $K[\,(x,y),\ {}^1\!/\!_2\,] = 1$ for all $(x,y) \in S$, and apply Lemma 6.1.

Thus $\partial_m S_M = \partial S_M = [0,1]$ but $\partial_P S_M = \{\,{}^1\!/\!_2\,\}$. That is, the Poisson boundary is the single point ${}^1\!/\!_2$.

The h-processes of $p[\,(x,y),\ (m,n)\,]$, and an urn model

In general, by (3.5),

$$p^h[\,(x,y),\ (m,n)\,] \;=\; \frac{1}{h(x,y)} p[\,(x,y),\ (m,n)\,]\, h(m,n)$$

and

$$\begin{aligned} p^h[\,(x,y),\ (x+1,y)\,] \;&=\; \frac{1}{h(x,y)}\frac{1}{2}\, h(x+1,y) \\ p^h[\,(x,y),\ (x,y+1)\,] \;&=\; \frac{1}{h(x,y)}\frac{1}{2}\, h(x,y+1) \end{aligned} \tag{6.8}$$

Thus if $h_\alpha(x,y) = K[\,(x,y),\ \alpha\,]$ in (6.4)

$$\begin{aligned} p^{h_\alpha}[\,(x,y),\ (x+1,y)\,] \;&=\; \alpha \\ p^{h_\alpha}[\,(x,y),\ (x,y+1)\,] \;&=\; 1-\alpha \end{aligned}$$

and by the strong law of large numbers

$$P_x^{h_\alpha}\left(\lim_{n\to\infty} \frac{X_n}{X_n+Y_n} = \alpha\right) \;=\; 1 \qquad \text{for all } (x,y) \in S, \quad 0 \le \alpha \le 1$$

This is equivalent to $P_x^{h_\alpha}\big((X_n,Y_n) \to \alpha\big) = 1$ by Theorem 6.1, which implies Theorem 5.1 in this case by the argument in Section 5.

The harmonic function $h(x,y)$ corresponding to $\mu_h(d\alpha) = d\alpha$ in (6.6) is

$$h(x,y) \;=\; \int_0^1 2^{x+y}\alpha^x(1-\alpha)^y\, d\alpha \;=\; 2^{x+y}\frac{x!\,y!}{(x+y+1)!}$$

and so by (6.8)

$$\begin{aligned} p^h[\,(x,y),\ (x+1,y)\,] \;&=\; \frac{x+1}{x+y+2} \\ p^h[\,(x,y),\ (x,y+1)\,] \;&=\; \frac{y+1}{x+y+2} \end{aligned}$$

This corresponds to the following urn model. Suppose that $(x,y) \in S$ means that there are $x+1$ red balls and $y+1$ green balls in an urn. Draw out a ball at random, and place it back in the urn along with another ball of the same color. Thus the probability of adding another red ball depends on the proportion of red balls in the urn at that time. While this proportion varies randomly, we have by Theorem 5.1 and Theorem 6.1

$$\begin{aligned} P_{(x,y)}^h\left(\lim_{n\to\infty} \frac{X_n}{X_n+Y_n} = Z_\infty \,\in\, d\alpha\right) \;&=\; \frac{1}{h(x,y)} K[\,(x,y),\ \alpha\,]\, d\alpha \\ &=\; \frac{(x+y+1)!}{x!\,y!}\alpha^x(1-\alpha)^y\, d\alpha \end{aligned}$$

which is a beta distribution in α with parameters $x+1$ and $y+1$. Since $X_n+Y_n = n+C$ by (6.2), $\lim_{n\to\infty} X_n/n$ exists with probability one. The limit, however, is not constant, but is a beta-distributed random variable whose parameters depend on X_0.

The Hausdorff moment problem

Let $\mu_0, \mu_1, \mu_2, \dots$ be a sequence of nonnegative numbers. A special case of the Hausdorff moment problem is to find conditions on μ_n such that

$$\mu_n \;=\; \int_0^1 t^n\,\mu(dt), \qquad n\ge 0 \tag{6.9}$$

for some measure $\mu(dt)\ge 0$ on $[0,1]$. Define $\Delta c_n = c_n - c_{n+1}$ ($n\ge 0$) for arbitrary sequences $\{c_n\}$. The classical result is that $\{\mu_n\}$ is of the form (6.9) if and only if $\Delta^a\mu_b \ge 0$ for all integers $a,b\ge 0$ (Widder, 1946). If (6.9) holds, then by induction

$$\Delta^a\mu_n \;=\; \int_0^1 t^n(1-t)^a\,\mu(dt) \;\ge\; 0$$

which proves the result one way.

For the other way, note that $\Delta^{a+1}\mu_n = \Delta^a\mu_n - \Delta^a\mu_{n+1}$, so that $\Delta^a\mu_n = \Delta^{a+1}\mu_n + \Delta^a\mu_{n+1}$. This implies that

$$h(x,y) \;=\; 2^{x+y}\Delta^x\mu_y \tag{6.10}$$

is p-harmonic on S in the sense of (6.5). Thus, if $\Delta^x\mu_y\ge 0$ for all x,y, then $h(x,y)$ in (6.10) is a nonnegative harmonic function in S, and the representation (6.9) follows from (6.6) and (6.10).

7. Harmonic Functions on an Abelian Group

Nonnegative harmonic functions of random walks on an Abelian group such as the integer lattice Z^d can be characterized in a nice way. Suppose that $p(x,y)$ is an irreducible transition function for a random walk on an (additive) Abelian group S. That is, for all $x,y,a\in S$,

$$p(x,y) \;=\; p(x+a,y+a) \;=\; p(0,y-x) \;=\; p(y-x) \tag{7.1}$$

where $p(y)=p(0,y)$. If $u(x)\ge 0$ is p-harmonic on S,

$$\begin{aligned} u(x) \;&=\; \sum_{y\in S} p(x,y)u(y) \;=\; \sum_{y\in S} p(0,y-x)\,u(y) \;=\; \sum_{y\in S} u(y)\,p(y-x)\\ &=\; \sum_{y\in S} u(x+y)\,p(y) \;=\; \sum_{y\in S} u(x+y)\,p_n(y) \end{aligned} \tag{7.2}$$

where $p_n(y)=p_n(0,y)$ is the n^{th} matrix power of $p(x,y)$. Thus, if S is Abelian, then any harmonic $u(x)$ on S can be expressed as a convex combination of translates of itself. These translates are also harmonic, since if $u(x)$ is p-harmonic

$$\sum_{y\in S} p(x,y)u(y+b) = \sum_{y\in S} p(y-x)u(y+b) = \sum_{y\in S} p(y-b-x)u(y) = u(x+b)$$

From this it follows

Theorem 7.1 (Choquet and Deny, 1960; Doob, Snell, and Williamson, 1960). *If $p(x,y)$ satisfies (7.1), then $u(x) \ge 0$ is minimal p-harmonic on Z^d with $u(0) = 1$ if and only if*

$$u(x) = e^{\alpha\cdot x} \qquad \text{for some } \alpha \in R^d \ \text{ with } \ \sum_{y\in S} e^{\alpha\cdot y} p(y) = 1$$

Corollary 7.1. *Set*

$$\Gamma_p \ = \ \{\, \alpha \in R^d : \sum_{y\in S} e^{\alpha\cdot y} p(y) = 1 \,\} \tag{7.3}$$

Then $u(x) \ge 0$ is p-harmonic if and only if

$$u(x) \ = \ \int_{\Gamma_p} e^{\alpha\cdot x}\, \mu_u(d\alpha) \qquad \text{for some measure } \mu_u(d\alpha) \ge 0 \tag{7.4}$$

Proof of Theorem 7.1. If $u(x) = e^{\alpha\cdot x}$ and $\alpha \in \Gamma_p$, then

$$\sum_{y\in S} u(x+y)p(y) \ = \ u(x)\sum_{y\in S} u(y)p(y) \ = \ u(x)$$

and $u(x)$ is harmonic. If $p(x,y)$ is recurrent, then all nonnegative harmonic functions are constant, and Theorem 7.1 is immediate. Hence it is sufficient to assume that $p(x,y)$ is transient, so that we can apply the results of Sections 1–5.

Assume that $u(x)$ is minimal p-harmonic with $u(0) = 1$. For any $y \in S$, there exists $n \ge 0$ such that $p_n(y) > 0$ by irreducibility, and $0 \le u(x+y) \le c_y u(x)$ by (7.2) with $c_y = 1/p_n(y)$. Thus $u(x+y) = d_y u(x)$ for all $x \in S$ by minimality. Setting $x = 0$ implies $d_y = u(y)$ and

$$u(x+y) \ = \ u(x)u(y) \qquad \text{for all } x, y \in S$$

Thus minimal nonnegative harmonic functions on an Abelian group are *multiplicative* in this sense.

By considering $u(x)$ for $x \in Z^d$ at basis elements in Z^d, we conclude that any multiplicative $u(x)$ must be of the form $u(x) = e^{\alpha\cdot x}$ for $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in R^d$. Since $u(x) = e^{\alpha\cdot x}$ is p-harmonic if and only if $\alpha \in \Gamma_p$, the representation (7.4) holds for any p-harmonic $u(x) \ge 0$ on Z^d.

Conversely, assume $u(x) = e^{\beta\cdot x}$ for some $\beta \in \Gamma_p$, and assume $0 \le h(x) \le Cu(x)$ for some p-harmonic $h(x)$. Then by (7.4)

$$0 \ \le \ \int_{\Gamma_p} e^{\alpha\cdot x}\, \mu_h(d\alpha) \ \le \ Ce^{\beta\cdot x} \qquad \text{for some } \ \mu_h(d\alpha) \ge 0$$

Set $x = (x_1, x_2, \dots, x_d)$ for $x_i \in Z^1$. Fixing x_j for $j \ne i$ for each i and letting $x_i \to \pm\infty$ then implies $\mathrm{Supp}(\mu_h) \subseteq \{\beta\}$ and hence $h(x) = cu(x)$. Thus $u(x)$ is minimal, which completes the proof of Theorem 7.1.

Theorem 7.1 has a number of interesting consequences for irreducible random walks on Z^d:

Corollary 7.2. *(i) Mean-zero irreducible random walks on Z^d have no nontrivial nonnegative harmonic functions. That is, if*

$$\sum_{y\in S} p(y)|y| < \infty \qquad \textit{and} \qquad \sum_{y\in S} p(y)y = 0 \in Z^d$$

then $\Gamma_p = \{0\}$ and all harmonic $u(x) \ge 0$ are constant.

(ii) $u(x) = 1$ is always minimal harmonic. That is, the Poisson boundary $\partial_P S_M$ is always $\{0\}$ while the minimal Martin boundary $\partial_m S_M = \Gamma_p$ in (7.3).

(iii) Suppose

$$\sum_{y\in S} p(y)|y|^n < \infty, \quad \textit{all } n \ge 0, \qquad \textit{but} \qquad \sum_{y\in S} p(y)e^{\alpha\cdot y} = \infty \quad \textit{for all } \alpha \ne 0$$

Then, $p(x,y)$ has no nontrivial nonnegative harmonic functions on Z^d.

Proof. (i) Let $\phi(y) = e^{\alpha\cdot y}$ for any $\alpha \in R^d$. Then by Jensen's inequality

$$1 \;=\; \phi\Big(\sum_{y\in S} p(y)y\Big) \;\le\; \sum_{y\in S} \phi(y)p(y) \tag{7.5}$$

with strict inequality in (7.5) unless $\phi(y)$ is constant on the support of $\{\,p(y)\,\}$. Thus by irreducibility $\sum_{y\in S} e^{\alpha\cdot y}p(y) > 1$ for $\alpha \ne 0$, and $\Gamma_p = \{0\}$.

(ii) Since $0 \in \Gamma_p$.

(iii) Since then $\Gamma_p = \{0\}$.

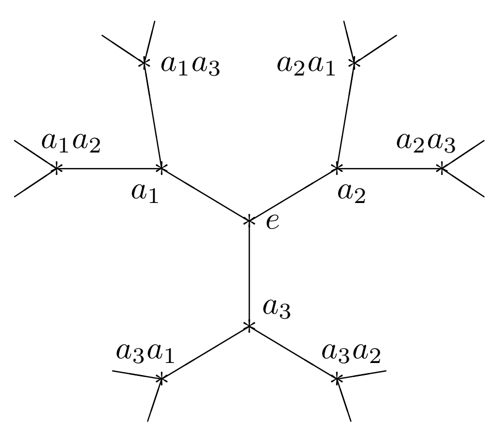
See Ney and Spitzer (1966), Spitzer (1976), and Woess (1994) for more information about irreducible random walks in Z^d.

8. Harmonic Functions on a Homogeneous Tree

Let T_r be an infinite homogeneous tree with $r \ge 3$ edges at each node. The tree T_r can be represented as the group G generated by r free involutions $\{a_1, a_2, \dots, a_r\}$. That is, $a_1^2 = a_2^2 = \dots = a_r^2 = e$ where e is the identity, but the a_i satisfy no other relations. Each $x \in G$ can then be written in a unique way in the form

$$x = a_{i_1}a_{i_2}\dots a_{i_d} \tag{8.1}$$

where $i_j \ne i_{j+1}$ $(1 \le j \le d-1)$ for some $d \ge 0$. The group G can be viewed as the set of all formal strings (8.1), where $x^{-1} = a_{i_d}a_{i_{d-1}}\dots a_{i_1}$ and the identity e is the empty string.



Neighborhood of e for $r = 3$

Let $|x| = d = d(x)$ be the length of the reduced word x in (8.1). Then $|e| = 0$

and $|xy| \le |x|\,|y|$ for $x, y \in G$. The tree T_r is the *Cayley graph* of the group G determined by the generators $\{a_1, \ldots, a_d\}$, which, by definition, is the graph whose nodes are the group elements $x \in G$ and whose edges are the pairs (x, xa_i) for $x \in G$ and generators a_i. The *graph distance* $d(x,y)$ is the number of edges in the shortest path between vertices $x, y \in T_r$. Thus $d(e,x) = |x| = d$ for x in (8.1), and $d(x,y) = |x^{-1}y|$ for $x, y \in G$. At each $x \in T_r$ (other than $x = e$) exactly one edge goes back towards e (i.e., $|xa_i| < |x|$, where a_i is the last letter in x,) and $r-1$ edges have $|xa_i| > |x|$.

Isotropic random walk on T_r

Nearest-neighbor isotropic random walk on T_r has the transition function

$$p(x, xa_i) \;=\; \frac{1}{r}, \qquad 1 \le i \le r \tag{8.2}$$

Thus $u(x)$ is p-harmonic on T_r if and only if $u(x)$ is the average of the values of u on the nearest neighbors of x:

$$u(x) \;=\; \frac{1}{r}\sum_{i=1}^{r} u(xa_i), \qquad x \in T_r$$

Let $X_n(w)$ be the random walk on T_r generated by (8.2) as in Section 2. By (8.2), the graph distance $Y_n = d(X_n, e) = |X_n|$ (which is the same as the length of the reduced word $X_n \in T_r$) is a Markov chain on $\{0, 1, 2, \ldots\}$ with transition function

$$\begin{aligned} q_{d,d+1} &= P(Y_n = d+1 \mid Y_{n-1} = d) \;=\; \frac{r-1}{r} && (d \ge 1) \\ q_{d,d-1} &= P(Y_n = d-1 \mid Y_{n-1} = d) \;=\; \frac{1}{r} && (d \ge 1) \\ q_{0,1} &= P(Y_n = 1 \mid Y_{n-1} = 0) \;=\; 1 \end{aligned} \tag{8.3}$$

The process Y_n acts like a random walk on Z^1 as long as $Y_n \ne e$. From this, it can be shown that

$$P_x\left(\lim_{n\to\infty} \frac{Y_n}{n} = \lim_{n\to\infty} \frac{d(X_n, e)}{n} = \frac{r-2}{r} > 0\right) \;=\; 1, \qquad \text{all } x \in T_r$$

(Sawyer, 1978; Sawyer and Steger, 1987). Thus $P_x(\lim_{n\to\infty} d(X_n, e) = \infty) = 1$ and X_n is transient for $r \ge 3$.

Define

$$\tau_y(w) = \min\{\, n \ge 0 : X_n(w) = y \,\}, \qquad y \in T_r$$

with $\tau_y(w) = \infty$ if $X_n(w) \ne y$ for $0 \le n < \infty$. By Lemma 5.1 (or by arguing directly from (2.2)), $P_x(\tau_y < \infty) = g(x,y)k(y)$ where $k(y) = P_y(X_n \ne y$ for all $n \ge 1)$. Since $\{X_n\}$ is transient, $k(y) = k > 0$. Thus

$$g(x,y) = P_x(\tau_y < \infty)\,/k, \qquad \text{all } x, y \in T_r$$

The Martin kernel $K(x,y)$ with reference point $x_0 = e$ is then

$$K(x,y) \;=\; \frac{g(x,y)}{g(e,y)} \;=\; \frac{P_x(\tau_y < \infty)}{P_e(\tau_y < \infty)} \;=\; \frac{P_{y^{-1}x}(\tau_e < \infty)}{P_{y^{-1}}(\tau_e < \infty)} \;=\; \frac{\phi(|y^{-1}x|)}{\phi(|y^{-1}|)} \tag{8.4}$$

where

$$\phi(d) \;=\; P(Y_n = 0 \text{ for some } n \ge 0 \mid Y_0 = d) \tag{8.5}$$

is the probability that $Y_n = d(X_n, e)$ ever attains the value 0 before being swept out to $+\infty$. The relations (8.3) imply that the probability (8.5) is the same as that for the classical *Gambler's Ruin* problem, whose solution we now derive.

Lemma 8.1. *The probabilities $\phi(d)$ in (8.5) satisfy $\phi(0) = 1$ and*

$$\phi(d) \;=\; \frac{1}{r}\phi(d-1) \;+\; \frac{r-1}{r}\phi(d+1) \qquad \text{for } d \geq 1 \tag{8.6}$$

Proof. If $d \geq 1$, then $\phi(d) = \sum_y q_{d,y}\phi(y)$ for $q_{d,y}$ in (8.3) by arguing as in the proof of Lemma 5.1. This implies Lemma 8.1.

Equation (8.6) is an example of what is called a *constant-coefficient linear difference equation*. It can be shown that a function $\psi(d)$ for $d \geq 0$ is a solution of (8.6) if and only if $\psi(d)$ is of the form

$$\psi(d) \;=\; C_1\,\lambda_1^d \;+\; C_2\,\lambda_2^d, \qquad \text{constants } C_1, C_2 \tag{8.7}$$

where λ_1, λ_2 are the two roots (assumed distinct) of the quadratic equation

$$\lambda \;=\; \frac{1}{r} + \frac{r-1}{r}\lambda^2 \tag{8.8}$$

This follows from the facts that (i) any function $\psi(d)$ in (8.7) is also a solution of (8.6), (ii) by induction, any solution of (8.6) is uniquely determined by $\phi(0)$ and $\phi(1)$, and (iii) constants C_1, C_2 can be chosen in (8.7) to match arbitrary initial values $\phi(0), \phi(1)$ in (8.6).

The quadratic equation (8.8) can be factored

$$(r-1)\lambda^2 - r\lambda + 1 \;=\; \big((r-1)\lambda - 1\big)\,(\lambda - 1) \;=\; 0$$

and the two roots are $\lambda_1 = 1$ and $\lambda_2 = 1/(r-1)$. Since $\lim_{d\to\infty}\phi(d) = 0$ by (8.5), and obviously $\phi(0) = 1$, it follows from (8.7) that

$$\phi(d) \;=\; P(Y_n = 0,\ \text{some } n \geq 0 \mid Y_0 = d) \;=\; \left(\frac{1}{r-1}\right)^d, \qquad d \geq 0$$

Since $Y_n = d(X_n, e)$,

$$P_x(\tau_e < \infty) \;=\; \left(\frac{1}{r-1}\right)^{|x|}$$

It then follows from (8.4) that

$$K(x,y) \;=\; \left(\frac{1}{q}\right)^{|x^{-1}y|-|y|}, \qquad q = r-1, \quad x, y \in T_r \tag{8.9}$$

The limiting behavior of $K(x,y)$ as $y \to \infty$ can be inferred from

Lemma 8.2. *For any sequence $\{\, y_n \,\} \subseteq T_r$,*

$$\lim_{n\to\infty} |x^{-1}y_n| - |y_n| \qquad \textit{exists for all } x \in T_r$$

if and only if $\lim_{n\to\infty} d_n = \lim_{n\to\infty} |y_n| = \infty$ *and, writing* $y_n = a_{i_1^n}\, a_{i_2^n}\, \ldots\, a_{i_{d_n}^n}$ *for* $1 \leq i_j^n \leq r$ $(i_{j+1}^n \neq i_j^n)$,

$$\lim_{n\to\infty} i_k^n = i_k \qquad \textit{exists for } 0 \leq k < \infty \tag{8.10}$$

Proof. Let

$$\begin{aligned} \phi_{i_1,i_2,\ldots,i_d} \;&=\; \lim_{n\to\infty} |(a_{i_1}a_{i_2}\ldots a_{i_d})^{-1}y_n| - |y_n|, \\ \widetilde{\phi}_d = \; &\min_{1\leq i_j\leq r, 1\leq j\leq d} \phi_{i_1,i_2,\ldots,i_d} \end{aligned} \tag{8.11}$$

Then $\widetilde{\phi}_d = -d$ by a diagonalization argument. The values i_k in Lemma 8.2 are the values that minimize $\phi_{i_1,i_2,\ldots,i_d}$ in (8.11).

The Ends of a Tree

The *space of ends* of the tree T_r is the set Ω of formal infinite reduced words

$$\omega = a_{i_1} a_{i_2} \dots a_{i_d} \dots \qquad (i_{j+1} \neq i_j) \tag{8.12}$$

Since $i_k \neq i_{k+1}$ in (8.10) by construction, the sequences $\{i_k\}$ arising in (8.10) can be identified with the formal infinite reduced words $\omega \in \Omega$. We have now shown the first part of

Theorem 8.1. *(i) The Martin boundary ∂S_M for isotropic nearest-neighbor random walk on T_r is the space of ends Ω of T_r, which is a compact metric space under the topology of convergence of each of the components a_i in (8.12). If a sequence $y_n \in T_r$ converges in the sense of (8.10) to a point $\omega \in \Omega$, then*

$$\lim_{n\to\infty} K(x, y_n) \;=\; K(x,\omega) = q^{-L(x,\omega)}$$

where if $x = a_{k_1} a_{k_2} \dots a_{k_m}$ and $\omega = a_{i_1} a_{i_2} \dots a_{i_n} \dots$ in (8.12)

$$\begin{aligned} L(x,\omega) &= |x| - 2j(x,\omega) \qquad \textit{for} \\ j(x,\omega) &= \max\{\, j : k_c = i_c \quad \textit{for } 1 \le c \le j \,\} \end{aligned} \tag{8.13}$$

(ii) A function $u(x) \ge 0$ is p-harmonic on T_r if and only if

$$u(x) \;=\; \int_\Omega q^{-L(x,\omega)}\, \mu_u(d\omega), \qquad \textit{some } \mu_u(d\omega) \ge 0 \tag{8.14}$$

Proof. *Part (i) follows from (8.9) and the proof of Lemma 8.2, and part (ii) follows from Theorem 4.1.*

Remarks. It follows from (8.13) and (8.14) that each $K(x,\omega)$ is minimal harmonic on T_r for $\omega \in \Omega$, so that $\partial_m S_M = \partial S_M = \Omega$. It can also be shown that, in contrast with the Abelian case, $\partial_P S_M = \partial_m S_M = \Omega$.

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PROF. STANLEY A. SAWYER
DEPARTMENT OF MATHEMATICS
WASHINGTON UNIVERSITY IN ST. LOUIS
ST. LOUIS, MO 63130, USA

Email address: `sawyer@math.wustl.edu`