Homework #2 Solutions

1.3

10. $\begin{bmatrix} 1 & 2 & x \\ 3 & -1 & 2 \end{bmatrix}\begin{bmatrix} y \\ x \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$

$y + 2x + x = 6 \Rightarrow y + 3x = 6$ $\quad$ $3y + 9x = 18$

$3y - x + 2 = 8 \Rightarrow 3y - x = 6$ $\quad$ $-(3y - x = 6)$

$10x = 12 \Rightarrow \boxed{\begin{matrix} x = 6/5 \\ y = 12/5 \end{matrix}}$

13. a) $\begin{bmatrix} -1 & 2 \\ 0 & 4 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ 2 & 5 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 12 \\ 8 & 20 \\ 19 & 19 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 6 & 3 \\ 9 & 6 \end{bmatrix} = \boxed{\begin{bmatrix} -2 & 12 \\ 2 & 17 \\ 10 & 13 \end{bmatrix}}$

b) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{bmatrix} - 2\begin{bmatrix} 3 & -2 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 14 & 8 \\ 16 & 9 \end{bmatrix} - \begin{bmatrix} 6 & -4 \\ 4 & 10 \end{bmatrix} = \boxed{\begin{bmatrix} 8 & 12 \\ 12 & -1 \end{bmatrix}}$

c) $\begin{bmatrix} -1 & 0 & 3 \\ 2 & 4 & 5 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 3 & -2 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 8 & 6 \\ 25 & 14 \end{bmatrix} + \begin{bmatrix} 3 & -2 \\ 2 & 5 \end{bmatrix} = \boxed{\begin{bmatrix} 11 & 4 \\ 27 & 19 \end{bmatrix}}$

d) not possible (2F is 3×2 and 3(AE) is 2×3)
matrices of unlike dimensions cannot be added or subtracted

e) not possible (BD is 3×2 and AE is 2×3)
matrices of unlike dimensions cannot be added or subtracted

19. $AB = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} -4 & 7 \\ 0 & 5 \end{bmatrix}$

$BA = \begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 9 & 2 \end{bmatrix}$

$\boxed{AB \neq BA}$

23. $A\vec{c} = \begin{bmatrix} 2 & -3 & 4 \\ -1 & 2 & 3 \\ 5 & -1 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 2(2) + 1(-3) + 4(4) \\ 2(-1) + 1(2) + 4(3) \\ 2(5) + 1(-1) + 4(-2) \end{bmatrix}$

$$\boxed{A\vec{c} = 2\begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} + 1\begin{bmatrix} -3 \\ 2 \\ -1 \end{bmatrix} + 4\begin{bmatrix} 4 \\ 3 \\ -2 \end{bmatrix}}$$

29. $A = \begin{bmatrix} -3 & 2 & 1 \\ 4 & 5 & 0 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}$

$A^T = \begin{bmatrix} -3 & 4 \\ 2 & 5 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} a_1^T \\ a_2^T \\ a_3^T \end{bmatrix}$

$$\boxed{A^TA = \begin{bmatrix} a_1^Ta_1 & a_1^Ta_2 & a_1^Ta_3 \\ a_2^Ta_1 & a_2^Ta_2 & a_2^Ta_3 \\ a_3^Ta_1 & a_3^Ta_2 & a_3^Ta_3 \end{bmatrix} = \begin{bmatrix} 25 & 14 & -3 \\ 14 & 29 & 2 \\ -3 & 2 & 1 \end{bmatrix}}$$

30. a) $\boxed{A = \begin{bmatrix} 2 & 3 & -3 & 1 & 1 \\ 3 & 0 & 2 & 0 & 3 \\ 2 & 3 & 0 & -4 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}}$

b.) $\boxed{\begin{bmatrix} 2 & 3 & -3 & 1 & 1 \\ 3 & 0 & 2 & 0 & 3 \\ 2 & 3 & 0 & -4 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \\ 3 \\ 5 \end{bmatrix}}$

c) $\boxed{\left[\begin{array}{ccccc|c} 2 & 3 & -3 & 1 & 1 & 7 \\ 3 & 0 & 2 & 0 & 3 & -2 \\ 2 & 3 & 0 & -4 & 0 & 3 \\ 0 & 0 & 1 & 1 & 1 & 5 \end{array}\right]}$

51. Let $C = [c_{ij}] = AA^T$

$$c_{ii} = \sum_{k=1}^{n} a_{ik} a^T_{ki} = \sum_{k=1}^{n} a_{ik} a_{ik} = \sum_{k=1}^{n} (a_{ik})^2$$

Thus if $AA^T = \mathbb{O}$, then $\sum_{k=1}^{n} (a_{ik})^2 = 0 \Rightarrow a_{ik} = 0$ for every i and k.

$$\boxed{\therefore A = \mathbb{O}}$$

1.4 6. Let $A = [a_{ij}]$, where $a_{ii} = k$ and $a_{ij} = 0$ if $i \neq k$, and let $B = [b_{ij}]$. Let $AB = C = [c_{ij}]$. Then if $i \neq j$,

$$c_{ij} = \sum_{l=1}^{n} a_{il} b_{lj} = k b_{ij}, \quad \text{if } i = j$$

$$c_{ii} = \sum_{l=1}^{n} a_{il} b_{li} = k b_{ii} \qquad \boxed{\therefore AB = kB}$$

20. $$A\vec{x} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = r \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \boxed{r = 3}$$

21. $$A\vec{x} = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix} \begin{bmatrix} -1/2 \\ 1/4 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1/2 \\ 2 \end{bmatrix} = r \begin{bmatrix} -1/2 \\ 1/4 \\ 1 \end{bmatrix} \Rightarrow \boxed{r = 2}$$

1.5 4. $$A + B = \begin{bmatrix} 3 & 2 & -1 \\ 0 & -4 & 3 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 6 & -3 & 2 \\ 0 & 2 & 4 \\ 0 & 0 & 3 \end{bmatrix} = \boxed{\begin{bmatrix} 9 & -1 & 1 \\ 0 & -2 & 7 \\ 0 & 0 & 3 \end{bmatrix} \rightarrow \text{upper triangular}}$$

$$AB = \begin{bmatrix} 3 & 2 & -1 \\ 0 & -4 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 6 & -3 & 2 \\ 0 & 2 & 4 \\ 0 & 0 & 3 \end{bmatrix} = \boxed{\begin{bmatrix} 18 & -5 & 11 \\ 0 & -8 & -7 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \text{upper triangular}}$$

7. a) $A^3 = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 1 \\ 3 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 1 \\ 3 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 1 \\ 3 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & -1 & -1 \\ 7 & 2 & -1 \\ 5 & 1 & -2 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 1 \\ 3 & 1 & 0 \end{bmatrix}$

$= \begin{bmatrix} -7 & -2 & 1 \\ 8 & 1 & -5 \\ 1 & -1 & -4 \end{bmatrix}$

$\boxed{A^3 = \begin{bmatrix} -7 & -2 & 1 \\ 8 & 1 & -5 \\ 1 & -1 & -4 \end{bmatrix}}$ ← (book is incorrect)

b) $B^2 = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \\ 2 & 0 & 3 \end{bmatrix}$ $\boxed{B^2 = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \\ 2 & 0 & 3 \end{bmatrix}}$

c) $(AB)^3 = \left(\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 1 \\ 3 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix}\right)^3 = \left(\begin{bmatrix} -2 & 0 & 0 \\ 1 & 1 & 4 \\ -1 & 1 & 4 \end{bmatrix}\right)^3$

$= \begin{bmatrix} -2 & 0 & 0 \\ 1 & 1 & 4 \\ -1 & 1 & 4 \end{bmatrix}\begin{bmatrix} -2 & 0 & 0 \\ 1 & 1 & 4 \\ -1 & 1 & 4 \end{bmatrix}\begin{bmatrix} -2 & 0 & 0 \\ 1 & 1 & 4 \\ -1 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ -5 & 5 & 20 \\ -1 & 5 & 20 \end{bmatrix}\begin{bmatrix} -2 & 0 & 0 \\ 1 & 1 & 4 \\ -1 & 1 & 4 \end{bmatrix}$

$= \begin{bmatrix} -8 & 0 & 0 \\ -5 & 25 & 100 \\ -13 & 25 & 100 \end{bmatrix}$ $\boxed{(AB)^3 = \begin{bmatrix} -8 & 0 & 0 \\ -5 & 25 & 100 \\ -13 & 25 & 100 \end{bmatrix}}$

12. Let $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $AB = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+2c & b+2d \\ c & d \end{bmatrix}$

$BA = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & 2a+b \\ c & 2c+d \end{bmatrix} = \begin{bmatrix} a+2c & b+2d \\ c & d \end{bmatrix}$

$a = a + 2c \Rightarrow c = 0$

$2a + b = b + 2d \Rightarrow a = d$

$B = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$ $a, b \in \mathbb{R}$

There are infinitely many of such matrices

13. For any $n\times n$ matrix A, $A^TA = AA^T$ is <u>false.</u>

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, then $A^TA = \begin{bmatrix} 10 & 14 \\ 14 & 20 \end{bmatrix}$ and $AA^T = \begin{bmatrix} 5 & 11 \\ 11 & 25 \end{bmatrix}$.

18. a) $(A + A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T$

$\Rightarrow \boxed{A + A^T \text{ is symmetric}}$

b.) $(A - A^T)^T = A^T - (A^T)^T = A^T - A = -(A - A^T)$

$\Rightarrow \boxed{A - A^T \text{ is skew symmetric}}$