Homework #4 Solutions

3.2 2. Show that P, the set of all polynomials, is a vector space.

Let $p(t)=\sum_{n=0}^{N} a_n t^n$, $q(t)=\sum_{n=0}^{N} b_n t^n$, and $r(t)=\sum_{n=0}^{N} c_n t^n$ be elements of P with $a_n, b_n, c_n \in \mathbb{R}$.

Let $j, k \in \mathbb{R}$.

a.) $p(t)\oplus q(t)=\sum_{n=0}^{N} a_n t^n+\sum_{n=0}^{N} b_n t^n=\sum_{n=0}^{N}(a_n+b_n)t^n \in P$ ✓

(1) $p(t)\oplus q(t)=\sum_{n=0}^{N} a_n t^n+\sum_{n=0}^{N} b_n t^n=\sum_{n=0}^{N} b_n t^n+\sum_{n=0}^{N} a_n t^n=q(t)\oplus p(t)$ ✓

(2) $p(t)\oplus(q(t)\oplus r(t))=\sum_{n=0}^{N} a_n t^n+\left(\sum_{n=0}^{N} b_n t^n+\sum_{n=0}^{N} c_n t^n\right)$

$=\left(\sum_{n=0}^{N} a_n t^n+\sum_{n=0}^{N} b_n t^n\right)+\sum_{n=0}^{N} c_n t^n=(p(t)\oplus q(t))\oplus r(t)$ ✓

(3) $\exists \vec{0}\in P \mid p(t)\oplus\vec{0}=\vec{0}\oplus p(t)=p(t)$ for $\forall p(t)\in P$

Let $\vec{0}=0$, then $p(t)\oplus 0=\sum_{n=0}^{N} a_n t^n+0=\sum_{n=0}^{N} a_n t^n=p(t)$ ✓

(4) $\exists -p(t)\in P, \forall p(t)\in P \mid p(t)\oplus -p(t)=-p(t)\oplus p(t)=\vec{0}$

Let $-p(t)=\sum_{n=0}^{N} -a_n t^n$, then $p(t)\oplus -p(t)=\sum_{n=0}^{N} a_n t^n+\sum_{n=0}^{N} -a_n t^n$

$=\sum_{n=0}^{N}(a_n-a_n)t^n=0=\vec{0}$ ✓

b.) $j\odot p(t)=j\sum_{n=0}^{N} a_n t^n=\sum_{n=0}^{N}(ja_n)t^n \in P$ ✓

(5) $j\odot(p(t)\oplus q(t))=j\odot\left(\sum_{n=0}^{N} a_n t^n+\sum_{n=0}^{N} b_n t^n\right)=j\sum_{n=0}^{N} a_n t^n+j\sum_{n=0}^{N} b_n t^n$

$=j\odot p(t)\oplus j\odot q(t)$ ✓

(6) $(j+k)\odot p(t)=(j+k)\sum_{n=0}^{N} a_n t^n=j\sum_{n=0}^{N} a_n t^n+k\sum_{n=0}^{N} a_n t^n$

$= j \odot p(t) \oplus k \odot p(t)$ ✓

(7) $j \odot (k \odot p(t)) = j \odot (k \sum_{n=0}^{N} a_n t^n) = jk \sum_{n=0}^{N} a_n t^n = (jk) \odot p(t)$ ✓

(8) $1 \odot p(t) = 1 \sum_{n=0}^{N} a_n t^n = \sum_{n=0}^{N} a_n t^n = p(t)$ ✓

4. Property 6 does not hold.

$(c+d) \odot (x,y) = (x, (c+d)y)$

$c \odot (x,y) \oplus d \odot (x,y) = (x, cy) \oplus (x, dy) = (2x, (c+d)y)$

$(x, (c+d)y) \neq (2x, (c+d)y)$

11. Let $y_1 \oplus y_2 = y_1 + y_2$ and $c \odot y_1 = cy_1$

a) $y_1 \oplus y_2 = y_1 + y_2$, $(y_1+y_2)' = y_1' + y_2'$, $(y_1+y_2)'' = y_1'' + y_2''$

$(y_1+y_2)'' - (y_1+y_2)' + 2(y_1+y_2) = (y_1'' - y_1' + 2y_1) + (y_2'' - y_2' + 2y_2) = 0$

$\Rightarrow y_1 \oplus y_2 \in V$ ✓

(1) $y_1 \oplus y_2 = y_1 + y_2 = y_2 + y_1 = y_2 \oplus y_1$ ✓

(2) $y_1 \oplus (y_2 \oplus y_3) = y_1 + (y_2 + y_3) = (y_1 + y_2) + y_3 = (y_1 \oplus y_2) \oplus y_3$ ✓

(3) Let $\vec{0} = 0$, $y_1 \oplus \vec{0} = y_1 + 0 = y_1$ ✓

(4) Let $-y_1 = -y_1$, $y_1 \oplus -y_1 = y_1 - y_1 = 0 = \vec{0}$ ✓

b) $c \odot y_1 = cy_1$, $(cy_1)' = cy_1'$, $(cy_1)'' = cy_1''$

$(cy_1)'' - (cy_1)' + 2(cy_1) = cy_1'' - cy_1' + 2cy_1 = c(y_1'' - y_1' + 2y_1) = 0$

$\Rightarrow c \odot y_1 \in V$ ✓

(5) $c \odot (y_1 \oplus y_2) = c(y_1 + y_2) = cy_1 + cy_2 = c \odot y_1 \oplus c \odot y_2$ ✓

(6) $(c+d) \odot y_1 = (c+d) y_1 = cy_1 + dy_1 = c \odot y_1 \oplus d \odot y_1$ ✓

(7) $c \odot (d \odot y_1) = c \odot (dy_1) = cdy_1 = (cd)y_1 = (cd) \odot y_1$ ✓

(8) $1 \odot y_1 = 1y_1 = y_1$ ✓

3.3 2. Yes, W is closed under $\oplus$ ($\vec{x}, \vec{y} \in \mathbb{R}^2 \Rightarrow \vec{x} + \vec{y} \in \mathbb{R}^2$) and closed under $\odot$ ($\vec{x} \in \mathbb{R}^2, c \in \mathbb{R} \Rightarrow c\vec{x} \in \mathbb{R}^2$)

4. No, take $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \notin W$.

W is not closed under vector addition or scalar multiplication

7. a) Not a subspace. Take $[b+2 \;\; b \;\; c \;\; d]$ and $[b'+2 \;\; b' \;\; c' \;\; d']$. $[b+2 \; b \; c \; d] + [b'+2 \; b' \; c' \; d'] = [b+b'+4 \;\; b+b' \;\; c+c' \;\; d+d$

$(b+b'+4) - (b+b') = 4 \neq 2$

Not closed under vector addition or scalar multiplication.

b.) Subspace.

Take $[a \;\; b \;\; a+2b \;\; a-3b]$ and $[a' \;\; b' \;\; a'+2b' \;\; a'-3b']$

$[a \;\; b \;\; a+2b \;\; a-3b] + [a' \;\; b' \;\; a'+2b' \;\; a'-3b']$

$= [(a+a') \;\; (b+b') \;\; (a+a')+2(b+b') \;\; (a+a')-3(b+b')]$

$\therefore$ Closed w.r.t vector addition

Take $n \in \mathbb{R}$ and $[a \;\; b \;\; a+2b \;\; a-3b]$

$n \cdot [a \;\; b \;\; a+2b \;\; a-3b] = [(na) \;\; (nb) \;\; (na)+2(nb) \;\; (na)-3(nb)]$

$\therefore$ Closed w.r.t scalar multiplication

c.) Subspace

Take $[0 \;\; -d \;\; c \;\; d]$ and $[0 \;\; -d' \;\; c' \;\; d']$

$[0 \;\; -d \;\; c \;\; d] + [0 \;\; -d' \;\; c' \;\; d'] = [0 \;\; -(d+d') \;\; (c+c') \;\; (d+d')]$

$\therefore$ Closed w.r.t vector addition

Take $n \in \mathbb{R}$ and $[0 \;\; -d \;\; c \;\; d]$

$n[0 \;\; -d \;\; c \;\; d] = [0 \;\; -(nd) \;\; (nc) \;\; (nd)]$

$\therefore$ Closed w.r.t scalar multiplicatio

14. a) Not a subspace, take $\begin{bmatrix}1&0\\0&0\end{bmatrix}$ and $\begin{bmatrix}0&0\\0&1\end{bmatrix}$, $\begin{bmatrix}1&0\\0&0\end{bmatrix}+\begin{bmatrix}0&0\\0&1\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix}$

b.) Subspace (the sum of two upper triangular matrices is upper triangular, and the scalar multiple of an upper triangular matrix is upper triangular)

c.) Subspace $((A+B)^T = A^T + B^T = -A + -B = -(A+B)$, $(cA)^T = cA^T = -(cA)$,

28. a) $[3 \;\; 6 \;\; 3 \;\; 0] = 3v_1$

b.) $\left[\begin{array}{cccc|c}1&4&1&-2&1\\2&1&2&3&0\\1&-2&6&-1&0\\0&3&-5&2&0\end{array}\right] \longrightarrow \left[\begin{array}{cccc|c}1&4&1&-2&1\\0&-7&0&7&-2\\0&-6&5&1&-1\\0&3&-5&2&0\end{array}\right] \longrightarrow \left[\begin{array}{cccc|c}1&4&1&-2&1\\0&-1&0&1&-2/7\\0&-3&0&3&-1\\0&3&-5&2&0\end{array}\right] \longrightarrow \left[\begin{array}{cccc|c}1&4&1&-2&1\\0&-1&0&1&-2/\\0&0&0&0&-1/\\0&3&-5&2&0\end{array}\right]$

Inconsistent

c.) $[3 \;\; 6 \;\; -2 \;\; 5] = v_1 + v_2 + v_4$

d) $\left[\begin{array}{cccc|c} 1 & 4 & 1 & -2 & 0 \\ 2 & 1 & 2 & 3 & 0 \\ 1 & -2 & 6 & -1 & 0 \\ 0 & 3 & -5 & 2 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 4 & 1 & -2 & 0 \\ 0 & -7 & 0 & 7 & 0 \\ 0 & -6 & 5 & 1 & 0 \\ 0 & 3 & -5 & 2 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 4 & 1 & -2 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & -3 & 0 & 3 & 1 \\ 0 & 3 & -5 & 2 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 4 & 1 & -2 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 3 & -5 & 2 & 1 \end{array}\right]$

Inconsistent

3.4 1. a) $[x\ y] = a_1[1\ 2] + a_2[-1\ 1]$

$x = a_1 - a_2 \quad y = 2a_1 + a_2 \implies a_1 = 1/3(x+y),\ a_2 = -2/3x + 1/3y$

$[1\ 2], [-1\ 1]$ Span $\mathbb{R}^2$

b.) $[0\ 0], [1\ 1], [-2\ -2]$ Do not span $\mathbb{R}^2$

$(x = a_2 - 2a_3 = y)$

c.) $[x, y] = a_1[1\ 3] + a_2[2\ -3] + a_3[0\ 2]$

$x = a_1 + 2a_2,\ y = 3a_1 - 3a_2 + 2a_3 \implies a_1 = 5/3x - 2/9y + 4/9r$

$a_2 = -1/3x + 1/9y - 2/9r,\ a_3 = r \quad r \in \mathbb{R}$

$[1\ 3], [2\ -3], [0\ 2]$ Span $\mathbb{R}^2$

d.) $[x\ y] = a_1[2\ 4] + a_2[-1\ 2]$

$x = 2a_1 - a_2 \quad y = 4a_1 + 2a_2 \implies a_1 = (2x+y)/8,\ a_2 = -1/2x + 1/4y$

$[2\ 4], [-1\ 2]$ Span $\mathbb{R}^2$

3. a.) Span $\mathbb{R}^4$ (a_1, a_2, a_3, and a_4 can be found for arbitrary $[x\ y\ z\ w]$)

b.) Do not Span $\mathbb{R}^4$ (a_1, a_2, and a_3 cannot be found)

c.) Do not Span $\mathbb{R}^5$ (a_1, a_2, a_3, a_4, and a_5 cannot be found)

d.) Span $\mathbb{R}^4$ (a_1, a_2, a_3, and a_4 can be found for arbitrary $[x\ y\ z\ w]$)

6. Yes, this set spans M_{22} (a_1, a_2, a_3, and a_4 can be found for arbitrary $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$)