Homework #5 Solutions

3.3 31. a) The line l_0 consists of all vectors of the form

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t\begin{bmatrix} u \\ v \\ w \end{bmatrix} \quad t, u, v, w \in \mathbb{R} \text{ with } u, v, w \text{ fixed}$$

a) $t_1\begin{bmatrix} u \\ v \\ w \end{bmatrix}, t_2\begin{bmatrix} u \\ v \\ w \end{bmatrix} \in l_0$

$$t_1\begin{bmatrix} u \\ v \\ w \end{bmatrix} \oplus t_2\begin{bmatrix} u \\ v \\ w \end{bmatrix} = (t_1+t_2)\begin{bmatrix} u \\ v \\ w \end{bmatrix} \in l_0 \checkmark$$

b) $c \odot t\begin{bmatrix} u \\ v \\ w \end{bmatrix} = ct\begin{bmatrix} u \\ v \\ w \end{bmatrix} \in l_0 \checkmark$

$\therefore$ l_0 is a subspace of $\mathbb{R}^3$ since it is closed w.r.t. vector addition and scalar multiplication

b.) The line l in $\mathbb{R}^3$ consists of all vectors of the form

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + t\begin{bmatrix} u \\ v \\ w \end{bmatrix} \quad x_0, y_0, z_0, t, u, v, w \in \mathbb{R} \text{ with } u, v, w, x_0, y_0, z_0 \text{ fixed}$$

a) $\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + t_1\begin{bmatrix} u \\ v \\ w \end{bmatrix}, \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + t\begin{bmatrix} u \\ v \\ w \end{bmatrix} \in l$

$$\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + t_1\begin{bmatrix} u \\ v \\ w \end{bmatrix} \oplus \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + t_2\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 2x_0 \\ 2y_0 \\ 2z_0 \end{bmatrix} + (t_1+t_2)\begin{bmatrix} u \\ v \\ w \end{bmatrix} \notin l$$

$\therefore$ l is not a subspace of $\mathbb{R}^3$ since it is not closed w.r.t. vector addition

3.4 7.

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 1 & 2 & 3 & 1 & 0 \\ 2 & 1 & 3 & 1 & 0 \\ 1 & 1 & 2 & 1 & 0 \end{array}\right] \xrightarrow[\substack{R_3-2R_1 \\ R_4-R_1}]{R_2-R_1} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{array}\right] \xrightarrow[R_2-R_3]{R_4-R_3} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_3-R_2} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 + x_3 = 0 \Rightarrow x_1 = -x_3$

$x_2 + x_3 = 0 \Rightarrow x_2 = -x_3$

$x_4 = 0$

$$x = r\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

$\left\{\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}\right\}$ spans the null space of A

8. $\left[\begin{array}{cccc|c} 1 & 1 & 2 & -1 & 0 \\ 2 & 3 & 6 & -2 & 0 \\ -2 & 1 & 2 & 2 & 0 \\ 0 & -2 & -4 & 0 & 0 \end{array}\right] \xrightarrow[R_3+2R_1]{R_2-2R_1} \left[\begin{array}{cccc|c} 1 & 1 & 2 & -1 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 3 & 6 & 0 & 0 \\ 0 & -2 & -4 & 0 & 0 \end{array}\right] \xrightarrow[\substack{R_3-3R_2 \\ R_4+2R_2}]{R_1-R_2} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

$x_1 - x_4 = 0 \Rightarrow x_1 = x_4$

$x_2 + 2x_3 = 0 \Rightarrow x_2 = -2x_3$

$x = r\begin{bmatrix}1\\0\\0\\1\end{bmatrix} + s\begin{bmatrix}0\\-2\\1\\0\end{bmatrix}$

$\boxed{\left\{\begin{bmatrix}1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\-2\\1\\0\end{bmatrix}\right\} \text{ spans the null space of } A}$

9. $\begin{bmatrix} 2 & 4 & 1 \\ -1 & -7 & 2 \\ 1 & -1 & 2 \end{bmatrix} \xrightarrow[\substack{R_2+R_1 \\ R_3-R_1}]{R_1+R_2} \begin{bmatrix} 1 & -3 & 3 \\ 0 & -10 & 5 \\ 0 & 2 & -1 \end{bmatrix} \xrightarrow{R_2+5R_3} \begin{bmatrix} 1 & -3 & 3 \\ 0 & 0 & 0 \\ 0 & 2 & -1 \end{bmatrix}$

$\boxed{\text{No, } \{x_1, x_2, x_3\} \text{ is not linearly independent}}$

10. $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 6 \\ 0 & -1 & 2 \\ 1 & 1 & 0 \end{bmatrix} \xrightarrow[R_4-R_1]{R_2-2R_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 4 \\ 0 & -1 & 2 \\ 0 & 0 & -1 \end{bmatrix} \xrightarrow[-1\cdot R_3]{R_2-2R_3} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$\boxed{\text{Yes, } \{x_1, x_2, x_3\} \text{ is linearly independent}}$

17. $\begin{bmatrix} 1 & 2 \\ 3 & c^2+2 \end{bmatrix} \xrightarrow{R_2-3R_1} \begin{bmatrix} 1 & 2 \\ 0 & c^2-4 \end{bmatrix}$ $\quad c^2-4 \neq 0 \Rightarrow c^2 \neq 2$

$\boxed{c \in \mathbb{R} \mid c \neq \pm 2}$

20. Suppose $a_1w_1 + a_2w_2 + a_3w_3 = a_1(v_1+v_2+v_3) + a_2(v_2+v_3) + a_3v_3 = 0$
Since $\{v_1, v_2, v_3\}$ is linearly independent, $a_1 = 0$,
$a_1 + a_2 = 0 \Rightarrow a_2 = 0$, $a_1 + a_2 + a_3 = 0 \Rightarrow a_3 = 0$.
Thus $\{w_1, w_2, w_3\}$ is linearly independent.

23. Suppose $\{v_1, v_2, v_3\}$ is linearly dependent. Then one of the v_i's is a linear combination of the preceding vectors in the list. It must be v_3 since

$\{v_1, v_2\}$ is linearly independent. Thus v_3 belongs to span $\{v_1, v_2\}$. But this is a contradiction. Thus $\{v_1, v_2, v_3\}$ is linearly independent.

26. Let $v_j = \sum_{i=1}^{k} a_{ij} u_i$. Then $w = \sum_{j=1}^{m} b_j v_j = \sum_{j=1}^{m} b_j \left[\sum_{i=1}^{k} a_{ij} u_i \right]$

Thus $w = \sum_{i=1}^{k} \left[\sum_{j=1}^{m} a_{ij} b_j \right] u_i$ and is a linear combination of the vectors in S.