Homework #6 Solutions

3.5 1. A basis for $\mathbb{R}^2$ is formed by <u>two</u>, <u>linearly independent</u> vectors.

a.) Yes $\left(\begin{bmatrix}1 & 1\\3 & -1\end{bmatrix} \to \begin{bmatrix}1 & 1\\0 & -4\end{bmatrix} \to \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right)$

b.) No (3 vectors)

c.) No (3 vectors)

d.) Yes $\left(\begin{bmatrix}1 & -2\\3 & 6\end{bmatrix} \longrightarrow \begin{bmatrix}1 & -2\\0 & 12\end{bmatrix} \longrightarrow \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right)$

2. A basis for $\mathbb{R}^3$ is formed by <u>three</u>, <u>linearly independent</u> vectors

a.) No (2 vectors)

b.) No (4 vectors)

c.) Yes $\left(\begin{bmatrix}3 & -1 & 0\\2 & 2 & 1\\2 & 1 & 0\end{bmatrix} \longrightarrow \begin{bmatrix}1 & -2 & 0\\2 & 2 & 1\\2 & 1 & 0\end{bmatrix} \longrightarrow \begin{bmatrix}1 & -2 & 0\\0 & 6 & 1\\0 & 5 & 0\end{bmatrix} \longrightarrow \begin{bmatrix}1 & -2 & 0\\0 & 1 & 0\\0 & 6 & 1\end{bmatrix} \to \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\right)$

d.) No (4 vectors)

11. $\begin{bmatrix}11\\10\\7\end{bmatrix} = 3\begin{bmatrix}3\\2\\1\end{bmatrix} + 2\begin{bmatrix}1\\2\\2\end{bmatrix}, \quad \begin{bmatrix}7\\6\\4\end{bmatrix} = 2\begin{bmatrix}3\\2\\1\end{bmatrix} + \begin{bmatrix}1\\2\\2\end{bmatrix}$

$\begin{bmatrix}3 & 1\\2 & 2\\1 & 2\end{bmatrix} \longrightarrow \begin{bmatrix}1 & 2\\0 & -2\\0 & -5\end{bmatrix} \longrightarrow \begin{bmatrix}1 & 0\\0 & 1\\0 & 0\end{bmatrix} \Rightarrow \begin{bmatrix}3\\2\\1\end{bmatrix}$ and $\begin{bmatrix}1\\2\\2\end{bmatrix}$ are linearly independent

$\boxed{\left\{\begin{bmatrix}3\\2\\1\end{bmatrix}, \begin{bmatrix}1\\2\\2\end{bmatrix}\right\} \text{ is a basis of } W \text{ and } \dim W = 2}$

20. a.) $\begin{bmatrix}0\\b\\c\end{bmatrix} = b\begin{bmatrix}0\\1\\0\end{bmatrix} + c\begin{bmatrix}0\\0\\1\end{bmatrix}$ $\boxed{\left\{\begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\} \text{ is a basis}}$

b.) $\begin{bmatrix}a+c\\a-b\\b+c\\-a+b\end{bmatrix} = (a-b)\begin{bmatrix}1\\1\\0\\-1\end{bmatrix} + (b+c)\begin{bmatrix}1\\0\\1\\0\end{bmatrix}$ $\boxed{\left\{\begin{bmatrix}1\\1\\0\\-1\end{bmatrix}, \begin{bmatrix}1\\0\\1\\0\end{bmatrix}\right\} \text{ is a basis}}$

c.) $\begin{bmatrix}b-5c\\b\\c\end{bmatrix} = b\begin{bmatrix}1\\1\\0\end{bmatrix} + c\begin{bmatrix}-5\\0\\1\end{bmatrix}$ $\boxed{\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}-5\\0\\1\end{bmatrix}\right\} \text{ is a basis}}$

28. a) A possible answer is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

b.) A possible answer is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

37. If $\dim V = n$, then V has a basis consisting of n vectors. Thus by Theorem 3.9, any $n+1$ vectors in V form a linearly dependent set.

3.6

3. $\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 2 & 1 & -1 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & -1 & -3 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & -2 & 0 & 0 \\ 0 & 1 & 3 & 1 & 0 \end{array}\right]$

$x_1 - 2x_3 = 0 \Rightarrow x_1 = 2x_3$

$x_2 + 3x_3 + x_4 = 0 \Rightarrow x_2 = -3x_3 - x_4$

$$\begin{bmatrix} 2x_3 \\ -3x_3 - x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

$\left\{ \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for the solution space of dimension 2

4. $\left[\begin{array}{ccccc|c} 1 & -1 & 1 & -2 & 1 & 0 \\ 3 & -3 & 2 & 0 & 2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc|c} 1 & -1 & 1 & -2 & 1 & 0 \\ 0 & 0 & -1 & 6 & -1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc|c} 1 & -1 & 0 & 4 & 0 & 0 \\ 0 & 0 & 1 & -6 & 1 & 0 \end{array}\right]$

$x_1 - x_2 + 4x_4 = 0 \Rightarrow x_1 = x_2 - 4x_4$

$x_3 - 6x_4 + x_5 = 0 \Rightarrow x_3 = 6x_4 - x_5$

$$\begin{bmatrix} x_2 - 4x_4 \\ x_2 \\ 6x_4 - x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 0 \\ 6 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 6 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for the solution space of dimension 3

3.8

1. $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} = 4/3 \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + 10/3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} -1 & 0 & 1 \\ -1 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 2 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 5 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

S spans $\mathbb{R}^3$; a basis for $\mathbb{R}^3$ is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

a) $\begin{bmatrix} 3 \\ 4 \\ 12 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 4 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 12 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

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