Homework #7 Solutions

3.8 13. a)

$$\begin{bmatrix} 1 & -1 & 2 & 3 \\ 2 & 6 & -8 & 1 \\ 5 & 3 & -2 & 10 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 8 & -12 & -5 \\ 0 & 8 & -12 & -5 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 1 & -3/2 & -5/8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 0 & 1/2 & 19/8 \\ 0 & 1 & -3/2 & -5/8 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \boxed{\text{row rank} = 2}$$

$x_1 = -1/2\, x_3 - 19/8\, x_4$

$x_2 = 3/2\, x_3 + 5/8\, x_4$

basis for null space: $\left\{ \begin{bmatrix} -1/2 \\ 3/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -19/8 \\ 5/8 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\boxed{\text{nullity} = 2}$

row rank + nullity = 2+2 = 4 = n. Theorem 3.18 holds ✓

b.)

$$\begin{bmatrix} 1 & 2 & 0 & 3 \\ 3 & 2 & -1 & 0 \\ 2 & -1 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & -4 & -1 & -9 \\ 0 & -5 & 0 & -5 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & -1 & -4 \\ 0 & 1 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 5 \end{bmatrix} \qquad \boxed{\text{row rank} = 3}$$

$x_1 = -x_4$

$x_2 = -x_4$

$x_3 = -5x_4$

basis for null space: $\left\{ \begin{bmatrix} -1 \\ -1 \\ -5 \\ 1 \end{bmatrix} \right\}$

$\boxed{\text{nullity} = 1}$

row rank + nullity = 3+1 = 4 = n. Theorem 3.18 holds ✓

17. a)

$$\begin{bmatrix} 1 & -2 & -3 & 4 \\ 4 & -1 & -5 & 6 \\ 2 & 3 & 1 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & -3 & 4 \\ 0 & 7 & 7 & -10 \\ 0 & 7 & 7 & -10 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & -3 & 4 \\ 0 & 7 & 7 & -10 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \boxed{\text{row rank} = 2}$$

$$\left[\begin{array}{cccc|c} 1 & -2 & -3 & 4 & 1 \\ 4 & -1 & -5 & 6 & 2 \\ 2 & 3 & 1 & -2 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & -2 & -3 & 4 & 1 \\ 0 & 7 & 7 & -10 & -2 \\ 0 & 7 & 7 & -10 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & -2 & -3 & 4 & 1 \\ 0 & 7 & 7 & -10 & -2 \\ 0 & 0 & 0 & 0 & 2 \end{array}\right] \qquad \boxed{\text{row rank} = 3}$$

$\boxed{\text{inconsistent}}$

b.)

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 5 & 1 & 5 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \\ 0 & -4 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \boxed{\text{row rank} = 2}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & -1 & 1 & 2 \\ 5 & 1 & 5 & 5 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -2 & 0 & -4 \\ 0 & -4 & 0 & -25 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & -17 \end{array}\right] \qquad \boxed{\text{row rank} = 3}$$

$\boxed{\text{inconsistent}}$

18. a)

$$\begin{bmatrix} 1 & 2 & -3 \\ -1 & 2 & 3 \\ 0 & 8 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & -3 \\ 0 & 4 & 0 \\ 0 & 8 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

row rank = 2 ≠ n = 3

$\boxed{\text{singular}}$

b.) $\begin{bmatrix} 1 & 2 & -3 \\ -1 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & -3 \\ 0 & 4 & 0 \\ 0 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ row rank $= 3 = n$ $\boxed{\text{nonsingular}}$

4.1 3. a) $\left\| \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right\| = \sqrt{0^2 + (-1)^2} = 1$ $\boxed{\|u - v\| = 1}$

b.) $\left\| \begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\| = \left\| \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\| = \sqrt{(-1)^2 + (1)^2} = \sqrt{2}$ $\boxed{\|u - v\| = \sqrt{2}}$

8. $(1/c)^2 + (2/c)^2 + (-2/c)^2 = 1/c^2 + 4/c^2 + 4/c^2 = 9/c^2 = 1$
$\Rightarrow c^2 = 9 \Rightarrow c = \pm 3$

$\boxed{c = \pm 3}$

25. $v \cdot w = 0$ $\begin{bmatrix} 2 \\ c \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = 2 - 2c + 3 = 0 \Rightarrow 2c = 5 \Rightarrow \boxed{c = 5/2}$

26. $v \cdot w = 0$ $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = a + 2b + c = 0$ $\Rightarrow b = 0, \; a = -c$
$v \cdot x = 0$ $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = a - b + c = 0$

$\boxed{\text{Yes, } v = \begin{bmatrix} a \\ 0 \\ -a \end{bmatrix} \; a \in \mathbb{R}}$

4.3 3. Define $(A, B) = \mathrm{Tr}(B^T A) = \sum_{ij} b^T_{ij} a_{ji} = \sum_{ij} b_{ji} a_{ji}$

a.) $(A, A) = \mathrm{Tr}(A^T A) = \sum_{ij} (a_{ji})^2 \geq .$ Also $(A, A) = 0$ iff $a_{ji} = 0$
$\Rightarrow A = 0$

b.) $(A, B) = \mathrm{Tr}(B^T A) = \sum_{ij} b_{ji} a_{ji} = \sum_{ij} a_{ji} b_{ji} = \mathrm{Tr}(A^T B) = (B, A)$

c.) $(A + B, C) = \mathrm{Tr}(C^T(A+B)) = \sum_{ij} c_{ji}(a_{ji} + b_{ji}) = \sum_{ij} c_{ji} a_{ji} + c_{ji} b_{ji}$

$= \sum_{ij} c_{ji} a_{ji} + \sum_{ij} c_{ji} b_{ji} = \mathrm{Tr}(C^T A) + \mathrm{Tr}(C^T B) = (A, C) + (B, C)$

d.) $(cA, B) = \mathrm{Tr}(B^T(cA)) = \mathrm{Tr}(c B^T A) = c\,\mathrm{Tr}(B^T A) = c(A, B)$

10. a) $(f,g) = \int_0^1 (1+t)(2-t)\,dt = \int_0^1 -t^2+t+2\,dt = -t^3/3 + t^2/2 + 2t \Big|_0^1$
$= -1/3 + 1/2 + 2 = 13/6$ $\boxed{(f,g) = 13/6}$

b.) $(f,g) = \int_0^1 1(3)\,dt = 3t\Big|_0^1 = 3$ $\boxed{(f,g) = 3}$

c.) $(f,g) = \int_0^1 1(3+2t)\,dt = 3t+t^2\Big|_0^1 = 4$ $\boxed{(f,g) = 4}$

16. $\|u+v\|^2 + \|u-v\|^2 = (u+v, u+v) + (u-v, u-v)$
$= (u,u) + 2(u,v) + (v,v) + (u,u) - 2(u,v) + (v,v)$
$= 2(u,u) + 2(v,v) = 2\|u\|^2 + 2\|v\|^2$
$\therefore \|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2$

29. a) $v_1 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, v_2 = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\|v_1\| = 1, \|v_2\| = 1, \|v_3\| = 1$
$v_1 \cdot v_2 = 0$ $v_1 \cdot v_3 = 0$ $v_2 \cdot v_3 = 0$
$\boxed{\text{This set is orthonormal}}$

b.) $v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\|v_1\| = \underline{\underline{\sqrt{2}}}, \|v_2\| = 1, \|v_3\| = 1$
$v_1 \cdot v_2 = 1$, $\underline{\underline{v_1 \cdot v_3 = 1}}$, $v_2 \cdot v_3 = 0$
$\boxed{\text{This set is neither}}$

c.) $v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $\|v_1\| = \underline{\underline{\sqrt{2}}}, \|v_2\| = \underline{\underline{\sqrt{2}}}, \|v_3\| = 1$
$\underline{\underline{v_1 \cdot v_2 = -1}}$, $v_1 \cdot v_3 = 0$, $\underline{\underline{v_2 \cdot v_3 = 1}}$
$\boxed{\text{This set is neither}}$

40. a) $w \in W \mid w \cdot u_1 = 0$ and $w \cdot u_2 = 0$
i.) $w_1 \oplus w_2 = w_1 + w_2$
$(w_1 + w_2) \cdot u_1 = w_1 \cdot u_1 + w_2 \cdot u_1 = 0 + 0 = 0$
$(w_1 + w_2) \cdot u_2 = w_1 \cdot u_2 + w_2 \cdot u_2 = 0 + 0 = 0$
$\Longrightarrow w_1 \oplus w_2 \in W$

ii.) $c \odot w_1 = cw_1$

$$(cw_1)\cdot u_1 = cw_1 \cdot u_1 = c(0) = 0$$
$$(cw_1)\cdot u_2 = cw_1 \cdot u_2 = c(0) = 0$$
$$\Longrightarrow c \odot w \in W$$

b.) $w = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ $\quad w \cdot u_1 = a + d = 0 \Longrightarrow a = -d$

$\quad w \cdot u_2 = b + d = 0 \Longrightarrow b = -d$

$$w = \begin{bmatrix} -d \\ -d \\ c \\ d \end{bmatrix} = d\begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix} + c\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$\boxed{\text{A possible basis: } \left\{ \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}}$