Homework #8 Solutions

4.4

1. a) $u_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad u_2 = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$

$v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$v_2 = \begin{bmatrix} -3 \\ 4 \end{bmatrix} - 5/5 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \end{bmatrix}$

$\boxed{\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \end{bmatrix} \right\}}$

b) $w_1 = v_1 / \|v_1\| = 1/\sqrt{5} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$

$w_2 = v_2 / \|v_2\| = 1/2\sqrt{5} \begin{bmatrix} -4 \\ 2 \end{bmatrix} = \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$

$\boxed{\left\{ \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}, \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} \right\}}$

11. $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 2 & 0 & 2 \\ 1 & 2 & 1 & 3 \end{bmatrix} \xrightarrow[R_3 - R_1]{R_2 - R_1} \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \quad \therefore \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$ is a basis for W

$u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad u_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/3 \\ -1/3 \\ 2/3 \end{bmatrix} \qquad v_2 = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$

$v_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \frac{6}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{3}{6} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \\ 0 \end{bmatrix} \qquad v_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

$w_1 = v_1 / \|v_1\| = v_1 / \sqrt{3} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$

$w_2 = v_2 / \|v_2\| = v_2 / \sqrt{6} = \begin{bmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}$

$w_3 = v_3 / \|v_3\| = v_3 / \sqrt{2} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$

$\boxed{\left\{ \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}, \begin{bmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}, \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} \right\}}$

14. $\begin{bmatrix} a \\ a+b \\ c \\ b+c \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} a + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} b + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} c$

$u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

$v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \\ 0 \\ 1 \end{bmatrix} \qquad v_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 2 \end{bmatrix}$

$v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{0}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \frac{2}{6} \begin{bmatrix} -1 \\ 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/3 \\ -1/3 \\ 1 \\ 1/3 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 1 \\ -1 \\ 3 \\ 1 \end{bmatrix}$

$w_1 = v_1 / \|v_1\| = v_1 / \sqrt{2} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix}, \quad w_2 = v_2 / \|v_2\| = v_2 / \sqrt{6} = \begin{bmatrix} -1/\sqrt{6} \\ 1/\sqrt{6} \\ 0 \\ 2/\sqrt{6} \end{bmatrix}$

$$w_3 = v_3/\|v_3\| = v_3/2\sqrt{3} = \begin{bmatrix} 1/2\sqrt{3} \\ -1/2\sqrt{3} \\ 3/2\sqrt{3} \\ 1/2\sqrt{3} \end{bmatrix}$$

$$\boxed{\left\{ \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/\sqrt{6} \\ 1/\sqrt{6} \\ 0 \\ 2/\sqrt{6} \end{bmatrix}, \begin{bmatrix} 1/2\sqrt{3} \\ -1/2\sqrt{3} \\ 3/2\sqrt{3} \\ 1/2\sqrt{3} \end{bmatrix} \right\}}$$

20. a) $u_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}^T, u_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}^T$

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}^T, v_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}^T - \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}^T = \begin{bmatrix} 1/2 \\ 1/2 \\ -1 \end{bmatrix}^T \quad v_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}^T$$

$$w_1 = v_1/\|v_1\| = v_1/\sqrt{2} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}^T, w_2 = v_2/\|v_2\| = v_2/\sqrt{6} = \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \end{bmatrix}^T$$

$$\boxed{\left\{ \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}^T, \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \end{bmatrix}^T \right\}}$$

b) $u = c_1 w_1 + c_2 w_2 = [5 \ -2 \ -3]$

$c_1 = (u, w_1)$, $c_2 = (u, w_2)$

$c_1 = 7/\sqrt{2}$, $c_2 = 9/\sqrt{6}$

$$\boxed{u = 7/\sqrt{2}\, w_1 + 9/\sqrt{6}\, w_2}$$

24. $u_1 = 1$, $u_2 = t$ $\quad u_3 = t^2$

$v_1 = 1$, $v_2 = t - \int_0^1 t\,dt / \int_0^1 1\,dt\ (1) = t - 1/2 / 1\ (1) = t - 1/2$

$v_3 = t^2 - \int_0^1 t^2 dt / \int_0^1 1\,dt\ (1) - \int_0^1 t^3 - \frac{t^2}{2} dt / \int_0^1 (t-1/2)^2 dt\ (t - 1/2)$

$\quad = t^2 - 1/3 / 1\ (1) - 1/12 / 1/12\ (t - 1/2) = t^2 - t + 1/6$

$w_1 = v_1/\|v_1\| = 1/1 = 1$

$w_2 = v_2/\|v_2\| = (t-1/2)/\sqrt{\int_0^1 (t-1/2)^2 dt} = (t-1/2)/\sqrt{\int_0^1 t^2 - t + 1/4\,dt}$

$\quad = (t-1/2)/\sqrt{1/3 - 1/2 + 1/4} = (t-1/2)/\sqrt{1/12} = \sqrt{12}(t-1/2)$

$w_3 = v_3/\|v_3\| = (t^2 - t + 1/6)/\sqrt{\int_0^1 (t^2 - t + 1/6)^2 dt}$

$\quad = (t^2 - t + 1/6)/\sqrt{1/180} = \sqrt{180}(t^2 - t + 1/6)$

$$\boxed{\left\{ 1, \sqrt{12}(t - 1/2), \sqrt{180}(t^2 - t + 1/6) \right\}}$$

4.5

1. a) $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} = 2a - 3b + c = 0 \Rightarrow c = 3b - 2a$

$$\boxed{\left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \right\}}$$

b.) $W^{\perp}$ is a plane with normal vector $w = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$

4. $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 0 \\ -1 \\ 3 \end{bmatrix} = 2a - c + 3d = 0$, $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 2 \\ -5 \end{bmatrix} = a + 2b + 2c - 5d = 0$

$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \\ 1 \\ -2 \end{bmatrix} = 3a + 2b + c - 2d = 0$, $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 2 \\ -1 \\ 4 \end{bmatrix} = 7a + 2b - c + 4d = 0$

$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & -5 & 0 \\ 2 & 0 & -1 & 3 & 0 \\ 3 & 2 & 1 & -2 & 0 \\ 7 & 2 & -1 & 4 & 0 \end{array}\right] \xrightarrow[R_4 - 3R_2]{R_3 - R_2} \left[\begin{array}{cccc|c} 1 & 2 & 2 & -5 & 0 \\ 2 & 0 & -1 & 3 & 0 \\ 1 & 2 & 2 & -5 & 0 \\ 1 & 2 & 2 & -5 & 0 \end{array}\right] \xrightarrow[R_4 - R_1,\ R_2 - 2R_1]{R_3 - R_1} \left[\begin{array}{cccc|c} 1 & 2 & 2 & -5 & 0 \\ 0 & -4 & -5 & 13 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$a + 2b + 2c - 5d = 0 \Rightarrow a = -2b - 2c + 5d$

$-4b - 5c + 13d = 0 \Rightarrow b = 1/4(-5c + 13d)$

$a = -1/2(-5c + 13d) - 2c + 5d$

$= 1/2\, c - 3/2\, d$

$$\boxed{\left\{ \begin{bmatrix} 1/2 \\ -5/4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3/2 \\ 13/4 \\ 0 \\ 1 \end{bmatrix} \right\}}$$

11. a) $\text{proj}_W v = 1/\sqrt{5}\, w_1 - 7/\sqrt{5}\, w_2$ $\boxed{\text{proj}_W v = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}}$

b.) $\text{proj}_W v = 8/\sqrt{5}\, w_1 - 1/\sqrt{5}\, w_2$ $\boxed{\text{proj}_W v = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}}$

c.) $\text{proj}_W v = -3/\sqrt{5}\, w_1 + 11/\sqrt{5}\, w_2$ $\boxed{\text{proj}_W v = \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}}$

13. a) $\text{proj}_W v = \int_{-\pi}^{\pi} t\,dt / \int_{-\pi}^{\pi} 1\,dt \,(1) + \int_{-\pi}^{\pi} t\cos t\,dt / \int_{-\pi}^{\pi} \cos^2 t\,dt \,(\cos t)$
$+ \int_{-\pi}^{\pi} t\sin t\,dt / \int_{-\pi}^{\pi} \sin^2 t\,dt \,(\sin t)$
$= 0(1) + 0(\cos t) + 2(\sin t)$

$$\boxed{\text{proj}_W v = 2\sin t}$$

b) $\text{proj}_W v = \int_{-\pi}^{\pi} t^2\,dt / \int_{-\pi}^{\pi} 1\,dt \,(1) + \int_{-\pi}^{\pi} t^2\cos t\,dt / \int_{-\pi}^{\pi} \cos^2 t\,dt \,(\cos t)$
$+ \int_{-\pi}^{\pi} t^2\sin t\,dt / \int_{-\pi}^{\pi} \sin^2 t\,dt \,(\sin t)$
$= \pi^2/3\,(1) + -4(\cos t) + 0(\sin t)$

$$\boxed{\text{proj}_W v = \pi^2/3 - 4\cos t}$$

c) $\text{proj}_W v = \int_{-\pi}^{\pi} e^t\,dt / \int_{-\pi}^{\pi} 1\,dt \,(1) + \int_{-\pi}^{\pi} e^t\cos(t)\,dt / \int_{-\pi}^{\pi} \cos^2 t\,dt \,(\cos t)$
$+ \int_{-\pi}^{\pi} e^t\sin t\,dt / \int_{-\pi}^{\pi} \sin^2 t\,dt \,(\sin t)$
$= \sinh(\pi)/\pi\,(1) + -\sinh(\pi)/\pi\,(\cos t) + \sinh(\pi)/\pi\,(\sin t)$

$$\boxed{\text{proj}_W v = (e^{\pi} - e^{-\pi})/2\pi\,(1 - \cos t + \sin t)}$$

15. $\text{proj}_W v = 2w_1 - 1/\sqrt{5}\,w_2 = \begin{bmatrix} -1/5 \\ 2 \\ -2/5 \end{bmatrix} = w$

$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = b = 0, \quad \begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{5} \\ 0 \\ 2/\sqrt{5} \end{bmatrix} = a/\sqrt{5} + 2c/\sqrt{5} = 0$
$\Rightarrow a = -2c$

basis for $W^{\perp}$: $\left\{ \begin{bmatrix} -2/\sqrt{5} \\ 0 \\ 1/\sqrt{5} \end{bmatrix} \right\}$

$\text{proj}_{W^{\perp}} v = -3/\sqrt{5}\; w_1^{\perp} = \begin{bmatrix} 6/5 \\ 0 \\ -3/5 \end{bmatrix} = u$

$$\boxed{w + u = \begin{bmatrix} -1/5 \\ 2 \\ -2/5 \end{bmatrix} + \begin{bmatrix} 6/5 \\ 0 \\ -3/5 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = v}$$

19. $d = \|v - \text{proj}_W v\| = \| [-1\ 0\ 1]^T - [1/5\ 0\ 2/5]^T \|$
$= \| [-6/5\ 0\ 3/5]^T \|$
$= \sqrt{(6/5)^2 + (3/5)^2}$

$\text{proj}_W v = 1/\sqrt{5}\; w_2 = \begin{bmatrix} 1/5 \\ 0 \\ 2/5 \end{bmatrix}$

$$\boxed{d = 3/\sqrt{5}}$$