Homework #11 Solutions

6.2 22.

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix} = (b-a)(c-a)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+a \end{vmatrix}$$

$$= (b-a)(c-a)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 0 & c-b \end{vmatrix} = (b-a)(c-a)\left[1 \cdot 1 \cdot (c-b)\right]$$

$$= (b-a)(c-a)(c-b)$$

$$\boxed{\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (b-a)(c-a)(c-b)}$$

6.3 1.

a) $\det(M_{13}) = \det\left(\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}\right) = 3(2) - 1(5) = 1$ $\quad\boxed{\det(M_{13}) = 1}$

b) $\det(M_{22}) = \det\left(\begin{bmatrix} 1 & -2 \\ 5 & -3 \end{bmatrix}\right) = 1(-3) - (-2)(5) = 7$ $\quad\boxed{\det(M_{22}) = 7}$

c) $\det(M_{31}) = \det\left(\begin{bmatrix} 0 & -2 \\ 1 & 4 \end{bmatrix}\right) = 0(4) - (-2)(1) = 2$ $\quad\boxed{\det(M_{31}) = 2}$

d) $\det(M_{32}) = \det\left(\begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}\right) = 1(4) - (-2)(3) = 10$ $\quad\boxed{\det(M_{32}) = 10}$

5. a) $\begin{vmatrix} 3 & 0 \\ 2 & 1 \end{vmatrix} = 3|1| - 0|2| = 3(1) = 3$ $\quad\boxed{\begin{vmatrix} 3 & 0 \\ 2 & 1 \end{vmatrix} = 3}$

d) $$\begin{vmatrix} 4 & 1 & 3 \\ 2 & 3 & 0 \\ 1 & 3 & 2 \end{vmatrix} = 4\begin{vmatrix} 3 & 0 \\ 3 & 2 \end{vmatrix} - 1\begin{vmatrix} 2 & 0 \\ 1 & 2 \end{vmatrix} + 3\begin{vmatrix} 2 & 3 \\ 1 & 3 \end{vmatrix}$$

$$= 4\left[3(2) - 0(3)\right] - 1\left[2(2) - 0(1)\right] + 3\left[2(3) - 3(1)\right]$$

$$= 4(6) - 1(4) + 3(3) = 29$$

$$\boxed{\begin{vmatrix} 4 & 1 & 3 \\ 2 & 3 & 0 \\ 1 & 3 & 2 \end{vmatrix} = 29}$$

e) $$\begin{vmatrix} 4 & 2 & 2 & 0 \\ 2 & 0 & 0 & 0 \\ 3 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix} = -0\begin{vmatrix} 2 & 0 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} + 0\begin{vmatrix} 4 & 2 & 2 \\ 3 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} - 1\begin{vmatrix} 4 & 2 & 2 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} + 0\begin{vmatrix} 4 & 2 & 2 \\ 2 & 0 & 0 \\ 3 & 0 & 0 \end{vmatrix}$$

$$= -1\left[0\begin{vmatrix} 2 & 2 \\ 0 & 0 \end{vmatrix} - 0\begin{vmatrix} 4 & 2 \\ 2 & 0 \end{vmatrix} + 1\begin{vmatrix} 4 & 2 \\ 2 & 0 \end{vmatrix}\right]$$

$$= -1(1)\left[4(0) - (2)(2)\right] = 4$$

$$\boxed{\begin{vmatrix} 4 & 2 & 2 & 0 \\ 2 & 0 & 0 & 0 \\ 3 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix} = 4}$$

11. a) $\begin{vmatrix} t-2 & 2 \\ 3 & t-3 \end{vmatrix} = (t-2)(t-3) - 2(3) = t^2 - 5t + 6 - 6 = t^2 - 5t$

$= t(t-5) = 0 \Rightarrow t = 0 \text{ or } 5$

$\boxed{t = 0 \text{ or } 5}$

b) $\begin{vmatrix} t-1 & -4 \\ 0 & t-4 \end{vmatrix} = (t-1)(t-4) - 0(-4) = (t-1)(t-4) = 0$

$\rightarrow t = 1 \text{ or } 4$

$\boxed{t = 1 \text{ or } 4}$

15. a) $\text{Area} = 1/2 \begin{vmatrix} 3 & 3 & 1 \\ -1 & -1 & 1 \\ 4 & 1 & 1 \end{vmatrix} = 1/2\left[3(-1-1) - 3(-1-4) + 1(-1+4)\right]$

$= 1/2\,[12] = 6$

$\boxed{\text{Area}_\Delta = 6}$

6.4

3. a) $A_{11} = \begin{vmatrix} 4 & 1 \\ -4 & 5 \end{vmatrix} = 24$, $A_{12} = -\begin{vmatrix} -3 & 1 \\ 4 & 5 \end{vmatrix} = 19$, $A_{13} = \begin{vmatrix} -3 & 4 \\ 4 & -4 \end{vmatrix} = -4$

$A_{21} = -\begin{vmatrix} 2 & 8 \\ -4 & 5 \end{vmatrix} = -42$, $A_{22} = \begin{vmatrix} 6 & 8 \\ 4 & 5 \end{vmatrix} = -2$, $A_{23} = -\begin{vmatrix} 6 & 2 \\ 4 & -4 \end{vmatrix} = 32$

$A_{31} = \begin{vmatrix} 2 & 8 \\ 4 & 1 \end{vmatrix} = -30$, $A_{32} = -\begin{vmatrix} 6 & 8 \\ -3 & 1 \end{vmatrix} = -30$, $A_{33} = \begin{vmatrix} 6 & 2 \\ -3 & 4 \end{vmatrix} = 30$

$\text{adj}A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$

$\boxed{\text{adj}A = \begin{bmatrix} 24 & -42 & -30 \\ 19 & -2 & -30 \\ -4 & 32 & 30 \end{bmatrix}}$

b) $\begin{vmatrix} 6 & 2 & 8 \\ -3 & 4 & 1 \\ 4 & -4 & 5 \end{vmatrix} = 6[20+4] - 2[-15-4] + 8[12-16]$

$= 6(24) - 2(-19) + 8(-4) = 150$

$\boxed{\det A = 150}$

c) $\begin{bmatrix} 6 & 2 & 8 \\ -3 & 4 & 1 \\ 4 & -4 & 5 \end{bmatrix} \cdot \begin{bmatrix} 24 & -42 & -30 \\ 19 & -2 & -30 \\ -4 & 32 & 30 \end{bmatrix} = \begin{bmatrix} 150 & 0 & 0 \\ 0 & 150 & 0 \\ 0 & 0 & 150 \end{bmatrix} = 150 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 24 & -42 & -30 \\ 19 & -2 & -30 \\ -4 & 32 & 30 \end{bmatrix} \cdot \begin{bmatrix} 6 & 2 & 8 \\ -3 & 4 & 1 \\ 4 & -4 & 5 \end{bmatrix} = \begin{bmatrix} 150 & 0 & 0 \\ 0 & 150 & 0 \\ 0 & 0 & 150 \end{bmatrix} = 150 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\boxed{\therefore A(\text{adj}A) = (\text{adj}A)A = \det(A) \cdot I_3}$

7. b) $\begin{vmatrix} 4 & 2 & 2 \\ 0 & 1 & 2 \\ 1 & 0 & 3 \end{vmatrix} = 4(3) - 2(-2) + 2(-1) = 14 = \det A$

$A_{11} = \begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} = 3$, $A_{21} = -\begin{vmatrix} 2 & 2 \\ 0 & 3 \end{vmatrix} = -6$, $A_{31} = \begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} = 2$

$A_{12} = -\begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix} = 2$, $A_{22} = \begin{vmatrix} 4 & 2 \\ 1 & 3 \end{vmatrix} = 10$, $A_{32} = -\begin{vmatrix} 4 & 2 \\ 0 & 2 \end{vmatrix} = -8$

$A_{13} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$, $A_{23} = -\begin{vmatrix} 4 & 2 \\ 1 & 0 \end{vmatrix} = 2$, $A_{33} = \begin{vmatrix} 4 & 2 \\ 0 & 1 \end{vmatrix} = 4$

$\text{adj}\, A = \begin{bmatrix} 3 & -6 & 2 \\ 2 & 10 & -8 \\ -1 & 2 & 4 \end{bmatrix}$ $A^{-1} = 1/\det A \cdot (\text{adj}\, A)$ $\boxed{A^{-1} = \begin{bmatrix} 3/14 & -3/7 & 1/7 \\ 1/7 & 5/7 & -4/7 \\ -1/14 & 1/7 & 2/7 \end{bmatrix}}$

c) $\begin{vmatrix} 3 & 2 \\ -3 & 4 \end{vmatrix} = 12 + 6 = 18 = \det A$

$A_{11} = 4$, $A_{21} = -(2) = -2$

$A_{12} = -(-3) = 3$, $A_{22} = 3$

$\text{adj}\, A = \begin{bmatrix} 4 & -2 \\ 3 & 3 \end{bmatrix}$ $A^{-1} = 1/\det A \cdot (\text{adj}\, A)$ $\boxed{A^{-1} = \begin{bmatrix} 2/9 & -1/9 \\ 1/6 & 1/6 \end{bmatrix}}$

13. If A is singular the $\det A = 0$. Then $A(\text{adj}\, A) = \det(A) I_n$ so $A(\text{adj}\, A) = \mathbb{O}$. If $\text{adj}\, A$ is nonsingular then $(\text{adj}\, A)^{-1}$ exists so $A(\text{adj}\, A)(\text{adj}\, A)^{-1} = \mathbb{O}(\text{adj}\, A)^{-1}$. Thus $A = \mathbb{O}$. If $A = \mathbb{O}$ then its adjoint must also be $\mathbb{O}$. Thus $\text{adj}\, A = \mathbb{O}$ so $\text{adj}\, A$ is singular. This is a contradiction.

$\boxed{\therefore \text{adj}\, A \text{ is singular}}$

14. If A is singular, then $\text{adj}\, A$ is also singular by Ex. 13 and $\det(\text{adj}\, A) = 0 = (\det A)^{n-1}$. If A is nonsingular, then $A(\text{adj}\, A) = \det(A) I_n$.

$\det(A) \det(\text{adj}\, A) = \det(\det(A) I_n) = (\det(A))^n$

Thus $\det(\text{adj}\, A) = [\det(A)]^{n-1}$.

$\boxed{\therefore \det(\text{adj}\, A) = [\det(A)]^{n-1}}$

6.5 3. $\begin{vmatrix} 2 & 1 & 0 \\ 2 & 0 & 1 \\ 3 & 2 & 3 \end{vmatrix} = 2(-2) - 1(3) + 0(4) = -7 \neq 0$

$\boxed{\text{Yes, } S \text{ is a linearly independent set of vectors in } \mathbb{R}^3}$

10. $\begin{vmatrix} 2 & 3 & 7 \\ -2 & 0 & -4 \\ 1 & 2 & 4 \end{vmatrix} = 2(8) - 3(-4) + 7(-4) = 0$

Since $\det A = 0$, we cannot solve the system by Cramer's Rule.

12. $\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & -2 \\ 0 & 1 & -1 & 0 \\ 1 & 3 & 2 & 3 \end{vmatrix} = 1 \begin{vmatrix} 2 & 0 & -2 \\ 1 & -1 & 0 \\ 3 & 2 & 3 \end{vmatrix} = 1[2(-3) - 0(3) - 2(5)] = -16 \neq 0$

Yes, S is a linearly independent set of matrices in M_{22}.