Homework #12 Solutions

7.1 2. $L(t) = 1$, $L(1) = t$ $\quad A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\quad \det(\lambda I_2 - A) = \det\left(\begin{bmatrix} \lambda & -1 \\ -1 & \lambda \end{bmatrix}\right)$

$p(\lambda) = \lambda^2 - 1 = (\lambda+1)(\lambda-1) = 0 \Rightarrow \lambda = 1 \text{ or } -1$

For $\lambda = 1$

$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ -1 & 1 & 0 \end{array}\right] \Rightarrow a = b$

For $\lambda = -1$

$\left[\begin{array}{cc|c} -1 & -1 & 0 \\ -1 & -1 & 0 \end{array}\right] \Rightarrow a = -b$

$$\boxed{\begin{array}{l} \lambda_1 = 1,\ \vec{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ \lambda_2 = -1,\ \vec{x}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \end{array}}$$

5. a) $p(\lambda) = \det\left(\begin{bmatrix} \lambda-2 & -1 \\ 1 & \lambda-3 \end{bmatrix}\right) = (\lambda-2)(\lambda-3)+1 = \boxed{\lambda^2 - 5\lambda + 7}$

b) $p(\lambda) = \det\left(\begin{bmatrix} \lambda-1 & -2 & -1 \\ 0 & \lambda-1 & -2 \\ 1 & -3 & \lambda-2 \end{bmatrix}\right) = (\lambda-1)^2(\lambda-2) + 4 + (\lambda-1) - 6(\lambda-1)$

$= \lambda^3 - 4\lambda^2 + 7$ $\quad \boxed{p(\lambda) = \lambda^3 - 4\lambda^2 + 7}$

c) $p(\lambda) = \det\left(\begin{bmatrix} \lambda-4 & 1 & -3 \\ 0 & \lambda-2 & -1 \\ 0 & 0 & \lambda-3 \end{bmatrix}\right) = (\lambda-4)(\lambda-2)(\lambda-3)$

$\boxed{p(\lambda) = \lambda^3 - 9\lambda^2 + 26\lambda - 24}$

d) $p(\lambda) = \det\left(\begin{bmatrix} \lambda-4 & -2 \\ -3 & \lambda-3 \end{bmatrix}\right) = (\lambda-4)(\lambda-3) - 6$

$\boxed{p(\lambda) = \lambda^2 - 7\lambda + 6}$

7. a) $p(\lambda) = \det\left(\begin{bmatrix} \lambda-1 & 1 \\ -2 & \lambda-4 \end{bmatrix}\right) = (\lambda-1)(\lambda-4) + 2 = (\lambda-2)(\lambda-3)$

For $\lambda_1 = 2$

$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ -2 & -2 & 0 \end{array}\right] \Rightarrow x = -y$

For $\lambda_2 = 3$

$\begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} \Rightarrow y = -2x$

$$\boxed{\begin{array}{l} p(\lambda) = \lambda^2 - 5\lambda + 6 \\ \lambda_1 = 2,\ \lambda_2 = 3 \end{array}}$$

$$\boxed{\vec{x}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix},\ \vec{x}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}}$$

b) $p(\lambda) = \det\left(\begin{bmatrix} \lambda-2 & 2 & -3 \\ 0 & \lambda-3 & 2 \\ 0 & 1 & \lambda-2 \end{bmatrix}\right) = (\lambda-2)^2(\lambda-3) - (\lambda-2)(2)$

$= (\lambda-2)(\lambda^2-5\lambda+4) = 0$

For $\lambda_1 = 1$

$\left[\begin{array}{ccc|c} -1 & 2 & -3 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \Rightarrow \begin{array}{l} y = z \\ x = -z \end{array} \quad \vec{x}_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

$$\boxed{\begin{array}{c} p(\lambda) = \lambda^3 - 7\lambda^2 + 14\lambda - 8 \\ \lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = 4 \end{array}}$$

For $\lambda_2 = 2$

$\left[\begin{array}{ccc|c} 0 & 2 & -3 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} y = 0 \\ z = 0 \end{array} \quad \vec{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$\boxed{\vec{x}_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix},\ \vec{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ \vec{x}_3 = \begin{bmatrix} 7 \\ -4 \\ 2 \end{bmatrix}}$$

For $\lambda_3 = 4$

$\left[\begin{array}{ccc|c} 2 & 2 & -3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \Rightarrow \begin{array}{l} y = -2z \\ x = 7/2\, z \end{array} \quad \vec{x}_3 = \begin{bmatrix} 7 \\ -4 \\ 2 \end{bmatrix}$

c) $p(\lambda) = \det\left(\begin{bmatrix} \lambda-2 & -2 & -3 \\ -1 & \lambda-2 & -1 \\ -2 & 2 & \lambda-1 \end{bmatrix}\right) = (\lambda-2)^2(\lambda-1) - 4 + 6$

$- 6(\lambda-2) - 2(\lambda-1) + 2(\lambda-2)$

$= \lambda^3 - 5\lambda^2 + 2\lambda + 8 = (\lambda+1)(\lambda-2)(\lambda-4) = 0$

For $\lambda = -1$

$\left[\begin{array}{ccc|c} -3 & -2 & -3 & 0 \\ -1 & -3 & -1 & 0 \\ -2 & 2 & -2 & 0 \end{array}\right] \quad \begin{array}{l} x = -z \\ y = 0 \end{array} \quad \vec{x}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$$\boxed{\begin{array}{c} p(\lambda) = \lambda^3 - 5\lambda^2 + 2\lambda + 8 \\ \lambda_1 = -1,\ \lambda_2 = 2,\ \lambda_3 = 4 \end{array}}$$

For $\lambda = 2$

$\left[\begin{array}{ccc|c} 0 & -2 & -3 & 0 \\ -1 & 0 & -1 & 0 \\ -2 & 2 & 1 & 0 \end{array}\right] \quad \begin{array}{l} x = -z \\ y = -3/2\, z \end{array} \quad \vec{x}_2 = \begin{bmatrix} -2 \\ -3 \\ 2 \end{bmatrix}$

$$\boxed{\vec{x}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix},\ \vec{x}_2 = \begin{bmatrix} -2 \\ -3 \\ 2 \end{bmatrix},\ \vec{x}_3 = \begin{bmatrix} 8 \\ 5 \\ 2 \end{bmatrix}}$$

For $\lambda = 4$

$\left[\begin{array}{ccc|c} 2 & -2 & -3 & 0 \\ -1 & 2 & -1 & 0 \\ -2 & 2 & 3 & 0 \end{array}\right] \quad \begin{array}{l} x = 4z \\ y = 5/2\, z \end{array} \quad \vec{x}_3 = \begin{bmatrix} 8 \\ 5 \\ 2 \end{bmatrix}$

d) $p(\lambda) = \det\left(\begin{bmatrix} \lambda+2 & 2 & -3 \\ 0 & \lambda-3 & 2 \\ 0 & 1 & \lambda-2 \end{bmatrix}\right) = (\lambda+2)(\lambda-3)(\lambda-2) - 2(\lambda+2)$

$= (\lambda+2)(\lambda^2 - 5\lambda + 4) = 0$

For $\lambda_1 = -2$

$\left[\begin{array}{ccc|c} 0 & 2 & -3 & 0 \\ 0 & -5 & 2 & 0 \\ 0 & 1 & -4 & 0 \end{array}\right] \quad \begin{array}{l} y = 0 \\ z = 0 \end{array} \quad \vec{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$\boxed{\begin{array}{c} p(\lambda) = \lambda^3 - 3\lambda^2 - 6\lambda + 8 \\ \lambda_1 = -2,\ \lambda_2 = 1,\ \lambda_3 = 4 \end{array}}$$

For $\lambda_2 = 1$

$\left[\begin{array}{ccc|c} 3 & 2 & -3 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \quad \begin{array}{l} y = z \\ x = 1/3\, z \end{array} \quad \vec{x}_2 = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}$

For $\lambda_3 = 4$

$\left[\begin{array}{ccc|c} 6 & 2 & -3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]$ $\begin{array}{l} y = -2z \\ x = 7/6\, z \end{array}$ $\vec{x}_3 = \begin{bmatrix} 7 \\ -12 \\ 6 \end{bmatrix}$

$\boxed{\vec{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{x}_2 = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}, \vec{x}_3 = \begin{bmatrix} 7 \\ -12 \\ 6 \end{bmatrix}}$

11. Let $A = [a_{ij}]$ be an $n \times n$ upper triangular matrix, that is, $a_{ij} = 0$ for $i > j$. Then the characteristic polynomial of A is $p(\lambda) = \det(\lambda I_n - A) = \begin{vmatrix} \lambda - a_{11} & -a_{12} & \cdots & -a_{1n} \\ 0 & \lambda - a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 \cdots & 0 & \lambda - a_{nn} \end{vmatrix}$

$p(\lambda) = (\lambda - a_{11})(\lambda - a_{22}) \ldots (\lambda - a_{nn})$, which we obtain by Theorem 6.7. Thus the eigenvalues of A are $a_{11}, \ldots, a_{nn}$, which are the elements on the main diagonal of A. A similar proof shows the same result if A is lower triangular.

14. Let V be an n-dimensional vector space and $L: V \rightarrow V$ be a linear operator. Let λ be an eigenvalue of L and W the subset of V consisting of the zero vector 0_V, and all the eigenvectors of L associated with λ. To show that W is a subspace of V, let $\vec{u}$ and $\vec{v}$ be eigenvectors of L corresponding to λ and let c_1 and c_2 be scalars. Thun $L(\vec{u}) = \lambda \vec{u}$ and $L(\vec{v}) = \lambda \vec{v}$. Therefore

$$L(c_1\vec{u} + c_2\vec{v}) = c_1 L(\vec{u}) + c_2 L(\vec{v}) = c_1 \lambda \vec{u} + c_2 \lambda \vec{v} = \lambda(c_1\vec{u} + c_2\vec{v})$$

Thus $c_1\vec{u} + c_2\vec{v}$ is an eigenvector of L with eigenvalue λ. Hence W is closed with respect to addition and scalar multiplication. Since technically an eigenvector is never zero we had to explicitly state that 0_V was in W since scalars c_1 and c_2 could be zero or $c_1 = -c_2$ and $\vec{u} = \vec{v}$ making the linear combination $c_1\vec{u} + c_2\vec{v} = 0_V$. It follows that W

is a subspace of V.

18. a.) $\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -2 & 0 & -2 & 0 \\ 2 & 0 & 2 & 0 \end{array}\right]$ $y=r$, $x=-z$ $\boxed{\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}}$

b.) $\begin{bmatrix} 2 & 2 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $x=0, y=0$, $w=0, z=r$ $\boxed{\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}}$

21. Proof by Induction: $A^k x = \lambda^k x$
Base Case: $Ax = \lambda x$ ✓ (Given)
Assume: $k=n$ is true; $A^n x = \lambda^n x$
Show: $k=n+1$ is true

$$A^n x = \lambda^n x$$
$$A(A^n x) = A(\lambda^n x)$$
$$A^{n+1} x = \lambda^n (Ax) = \lambda^n (\lambda x) = \lambda^{n+1} x$$

$$\boxed{\therefore A^k x = \lambda^k x \text{ if } Ax = \lambda x}$$

7.2 1. $L(t^2) = 2t$, $L(t) = 1$, $L(1) = 0$

$A = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ $p(\lambda) = \det\left(\begin{bmatrix} \lambda & 0 & 0 \\ -2 & \lambda & 0 \\ 0 & -1 & \lambda \end{bmatrix}\right) = \lambda^3 = 0$

$\lambda = 0$ (mult. 3)

For $\lambda = 0$

$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \Rightarrow x=0$, $y=0$ $\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

There is only one linearly independent eigenvector. Therefore by Theorem 7.4 L is not diagonalizable.

5. $L(t^2) = 2t^2$, $L(t) = t^2 - 3t$, $L(1) = t^2 + 2t + 4$

$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -3 & 2 \\ 0 & 0 & 4 \end{bmatrix}$ $p(\lambda) = (\lambda - 2)(\lambda + 3)(\lambda - 4)$

$$\boxed{\lambda_1 = -3, \lambda_2 = 2, \lambda_3 = 4}$$

For $\lambda_1 = -3$

$$\left[\begin{array}{ccc|c} -5 & -1 & -1 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & -7 & 0 \end{array}\right] \quad \begin{array}{l} c=0 \\ b=-5a \end{array} \quad [\vec{x}_1]_S = \begin{bmatrix} 1 \\ -5 \\ 0 \end{bmatrix} \Rightarrow \quad \vec{x}_1 = t^2 - 5t$$

For $\lambda_2 = 2$

$$\left[\begin{array}{ccc|c} 0 & -1 & -1 & 0 \\ 0 & 5 & -2 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right] \quad \begin{array}{l} c=0 \\ b=0 \end{array} \quad [\vec{x}_2]_S = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \quad \vec{x}_2 = t^2$$

For $\lambda_3 = 4$

$$\left[\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ 0 & 7 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} b = 2/7\, c \\ a = 9/14\, c \end{array} \quad [\vec{x}_3]_3 = \begin{bmatrix} 9 \\ 4 \\ 14 \end{bmatrix} \Rightarrow \quad \vec{x}_3 = 9t^2 + 4t + 14$$

$\vec{x}_1 = t^2 - 5t$, $\vec{x}_2 = t^2$, $\vec{x}_3 = 9t^2 + 4t + 14$
Yes L is diagonalizable by Theorems 7.3, 7.4, 7.5

8. Let $D = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$ and $P = \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}$. Then $P^{-1}AP = D$,
so $A = PDP^{-1} = \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} -1/3 & 1/3 \\ 2/3 & 1/3 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} -2/3 & 2/3 \\ -2 & -1 \end{bmatrix}$

$A = \begin{bmatrix} -4/3 & -5/3 \\ -10/3 & 1/3 \end{bmatrix}$ is a matrix whose eigenvalues and associated eigenvectors are as given.

14. Let A be the given matrix.
a) Since A is upper triangular its eigenvalues are its diagonal entries. Since $\lambda = 2$ is an eigenvalue of multiplicity 2 we must show, by Theorem 7.4, that it has two linearly independent eigenvectors.

$$(2I_3 - A)x = \begin{bmatrix} 0 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} x = 0 \quad \Rightarrow y = 0,\ x = r,\ z = s$$

It follows that there are two arbitrary constants in the general solution so there are two linearly independent eigenvectors. Hence the matrix is diagonalizable.

b.) Since A is upper triangular its eigenvalues are its diagonal entries. Since $\lambda = 2$ is an eigenvalue of multiplicity 2 we must show it has two linearly independent eigenvectors using Theorem 7.4.

$$(2I_3 - A)x = \begin{bmatrix} 0 & -3 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} x = 0 \Rightarrow x = r, y = 0, z = 0$$

It follows that there is only one arbitrary constant in the general solution so that there is only one linearly independent eigenvector. Hence the matrix is not diagonalizable.

c.) The matrix is lower triangular hence its eigenvalues are its diagonal entries. Since they are distinct the matrix is diagonalizabl

d.) The eigenvalues of A are $\lambda_1 = 0$ with associated eigenvector $x_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, and $\lambda_2 = \lambda_3 = 3$, with associated eigenvector $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Since there are not two linearly independent eigenvectors associated with $\lambda_2 = \lambda_3 = 3$, A is not diagonalizabl

24. If A is diagonalizable, then there is a nonsingular matrix P so that $P^{-1}AP = D$, a diagonal matrix. Then $A = PD'P^{-1}$ so $A^{-1} = (PD P^{-1})^{-1}$
$A^{-1} = (P^{-1})^{-1} D^{-1} P^{-1} = PD^{-1}P^{-1}$. Since D^{-1} is a diagonal matrix, we conclude that A^{-1} is diagonalizable.

26. $(AB^{-1}) = AB^{-1}(AA^{-1}) = A(B^{-1}A)A^{-1}$
So AB^{-1} and $B^{-1}A$ are similar and thus have the same eigenvalues by Theorem 7.2

7.3 2. $p(\lambda) = \det\left(\begin{bmatrix} \lambda & -4 & 0 \\ -1/4 & \lambda & 0 \\ 0 & -1/2 & \lambda \end{bmatrix}\right) = \lambda^3 - \lambda = \lambda(\lambda+1)(\lambda-1)$

For $\lambda = 1$
$\left[\begin{array}{ccc|c} 1 & -4 & 0 & 0 \\ -1/4 & 1 & 0 & 0 \\ 0 & -1/2 & 1 & 0 \end{array}\right]$ $y = 2z$ $x = 4y$ $\vec{x} = \begin{bmatrix} 8 \\ 2 \\ 1 \end{bmatrix}$

$\boxed{\text{A possible stable age distribution is } \begin{bmatrix} 8 \\ 2 \\ 1 \end{bmatrix}.}$