7.4

Homework #13 Solutions

1. $P^T = \begin{bmatrix} 2/3 & 2/3 & 1/3 \\ -2/3 & 1/3 & 2/3 \\ 1/3 & -2/3 & 2/3 \end{bmatrix}$

$$P^TP = \begin{bmatrix} 2/3 & 2/3 & 1/3 \\ -2/3 & 1/3 & 2/3 \\ 1/3 & -2/3 & 2/3 \end{bmatrix}\begin{bmatrix} 2/3 & -2/3 & 1/3 \\ 2/3 & 1/3 & -2/3 \\ 1/3 & 2/3 & 2/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\boxed{\therefore \text{ Since } P^TP = I_3,\ P \text{ is orthogonal}}$

14. If $Ax = \lambda x$, then $(P^{-1}AP)P^{-1}x = P^{-1}(\lambda x) = \lambda(P^{-1}x)$, so that $B(P^{-1}x) = \lambda(P^{-1}x)$

$\boxed{\therefore B(P^{-1}x) = \lambda(P^{-1}x)}$

16. $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ $p(\lambda) = \det(\lambda I_3 - A) = \det\left(\begin{bmatrix} \lambda & 0 & -1 \\ 0 & \lambda & 0 \\ -1 & 0 & \lambda \end{bmatrix}\right)$

$$p(\lambda) = \lambda^3 - \lambda = \lambda(\lambda+1)(\lambda-1)$$

$\lambda_1 = -1,\ \lambda_2 = 0,\ \lambda_3 = 1$

For $\lambda = -1$

$\left[\begin{array}{ccc|c} -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & -1 & 0 \end{array}\right] \longrightarrow \vec{x}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

For $\lambda = 0$

$\left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right] \longrightarrow \vec{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

For $\lambda = 1$

$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{array}\right] \longrightarrow \vec{x}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$\vec{x}_1$, $\vec{x}_2$, and $\vec{x}_3$ are orthogonal (i.e $\vec{x}_1 \cdot \vec{x}_2 = 0$, etc)

$\vec{x}_1/\|x_1\| = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$, $\vec{x}_2/\|x_2\| = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{x}_3/\|x_3\| = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$

$$\boxed{A \text{ is similar to } D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } P = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}}$$

27. $A = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & 2 \\ 2 & 2 & 2 \end{bmatrix}$ $p(\lambda) = \det(\lambda I_3 - A) = \det\left(\begin{bmatrix} \lambda-1 & 1 & -2 \\ 1 & \lambda-1 & -2 \\ -2 & -2 & \lambda-2 \end{bmatrix}\right)$

$p(\lambda) = (\lambda-1)^2(\lambda-2) + 4 + 4 - 4(\lambda-1) - 4(\lambda-1) - (\lambda-2)$

$p(\lambda) = \lambda^3 - 4\lambda^2 - 4\lambda + 16 = (\lambda-2)(\lambda+2)(\lambda-4)$

$\lambda_1 = 2, \lambda_2 = -2, \lambda_3 = 4$

$\boxed{A \text{ is similar to } D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}}$

30. If L is orthogonal, then $\|L(v)\| = \|v\|$ for any v in V. If λ is an eigenvalue of L then $L(x) = \lambda x$, so $\|L(x)\| = \|\lambda x\|$, which implies that $\|\lambda x\| = \|x\|$. Then $\|\lambda x\| = \sqrt{(\lambda x, \lambda x)} = \sqrt{\lambda^2 (x,x)} = |\lambda|\sqrt{(x,x)} = |\lambda|\,\|x\|$. Thus $|\lambda|\,\|x\| = \|x\|$, and since x is an eigenvector, it cannot be the zero vector, so $|\lambda| = 1$.

7.6

1. a.) $\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} -3 & 5/2 \\ 5/2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

b.) $\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 2 & 3/2 & -5/2 \\ 3/2 & 0 & 7/2 \\ -5/2 & 7/2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

c.) $\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 3 & 1/2 & -1/2 \\ 1/2 & 1 & -2 \\ -1/2 & -2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

7. $2x_1x_3 = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ (From 7.4 #16 $\lambda_1 = 1, \lambda_2 = 0, \lambda_3 = -1$)

$\boxed{\text{So } 2x_1x_3 = y_1^2 - y_3^2}$

17. $g(x) = 4x_2^2 + 4x_3^2 - 10x_2x_3 = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 & -5 \\ 0 & -5 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$p(\lambda) = \det\left(\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda-4 & 5 \\ 0 & 5 & \lambda-4 \end{bmatrix}\right) = \lambda(\lambda-4)^2 - 25\lambda = \lambda(\lambda^2 - 8\lambda - 9)$
$= \lambda(\lambda-9)(\lambda+1)$

$\lambda_1 = 9, \lambda_2 = -1, \lambda_3 = 0$

Let $H = \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad D_1 = H^T \begin{bmatrix} 9 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$p = 1$
$s = 2(1) - 2 = 0$

$\boxed{g(x) = y_1^2 - y_2^2,\ \text{rank} = 2 \quad \text{signature} = 0}$

23. a) $p(\lambda) = (\lambda-2)^2 - 1 = \lambda^2 - 4\lambda + 3 = (\lambda-3)(\lambda-1) \quad \lambda = 1, 3$

$\boxed{\therefore \text{positive definite}}$

b) $p(\lambda) = (\lambda-2)^2 - 1 = (\lambda-3)(\lambda-1),\ \lambda = 1, 3$

$\boxed{\therefore \text{positive definite}}$

c) $p(\lambda) = (\lambda-3)^3 - (\lambda-3) = (\lambda-3)(\lambda^2 - 6\lambda + 8) = (\lambda-3)(\lambda-4)(\lambda-2)$
$\lambda = 2, 3, 4$

$\boxed{\therefore \text{positive definite}}$

d) $p(\lambda) = (\lambda-1)(\lambda-2)(\lambda+3) \quad \lambda = 1, 2, -3$ $\boxed{\therefore \text{not positive definite}}$

e) $p(\lambda) = (\lambda-2)^2 - 4 = \lambda^2 - 4\lambda = \lambda(\lambda-4) \quad \lambda = 0, 4$

$\boxed{\therefore \text{not positive definite}}$

25. $(P^TAP)^T = P^TA^TP = P^TAP$ since $A^T = A$

$\boxed{\therefore (P^TAP)^T = P^TAP \text{ so } P^TAP \text{ is symmetric.}}$

7.9 13. $\bar{x}_1 = 1/5 \sum_{j=1}^{5} x_{j1} = 1/5\,(810 + 841 + 1094 + 532 + 890) = 833.4$

$\bar{x}_2 = 1/5 \sum_{j=1}^{5} x_{j2} = 1/5\,(715 + 1124 + 1426 + 685 + 468) = 883.6$

$$\boxed{\text{Sample Means} \approx \begin{bmatrix} 833.4 \\ 883.6 \end{bmatrix}}$$

$$s_{11} = s_1^2 = 1/5 \sum_{j=1}^{5} (x_{j1} - \bar{x}_1)^2 = 1/5\Big[(810 - 833.4)^2 + (841 - 833.4)^2 + (1094 - 833.4)^2 + (532 - 833.4)^2 + (890 - 833.4)^2\Big]$$
$$= 32512.64$$

$$s_{12} = s_{21} = 1/5 \sum_{j=1}^{5} (x_{j1} - \bar{x}_1)(x_{j2} - \bar{x}_2) = 36691.36$$

$$S_{22} = S_2^2 = 1/5 \sum_{j=1}^{5} (x_{j2} - \bar{x}_2)^2 = 1/5 \left[(715 - 883.6)^2 + (1124 - 883.6)^2 + (1426 - 883.6)^2 + (685 - 883.6)^2 + (468 - 883.6)^2 \right]$$

$$= 118516.24$$

$$\boxed{\text{Covariance Matrix} \approx \begin{bmatrix} 32512.64 & 36691.36 \\ 36691.36 & 118516.24 \end{bmatrix}}$$

$$p(\lambda) = (\lambda - 32512.64)(\lambda - 118516.24) - 36691.36^2$$

$$\lambda \cong 132042.4 \text{ or } 18986.5$$

$$\left[\begin{array}{cc|c} 99529.8 & -36691.36 & 0 \\ -36691.36 & 13526.2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 2.7126 & -1 & 0 \\ -2.7126 & 1 & 0 \end{array}\right]$$

$$U_1 = \begin{bmatrix} 2.7126/\sqrt{2.716^2+1^2} \\ 1/\sqrt{2.716^2+1^2} \end{bmatrix} = \begin{bmatrix} .938 \\ .346 \end{bmatrix}$$

$$Y_1 = \begin{bmatrix} 810 & 715 \\ 841 & 1124 \\ 1094 & 1426 \\ 532 & 685 \\ 890 & 468 \end{bmatrix} \begin{bmatrix} .938 \\ .346 \end{bmatrix} = \begin{bmatrix} 1007.3 \\ 1177.9 \\ 1519.7 \\ 736.1 \\ 996.9 \end{bmatrix}$$

$$\boxed{\text{First Principal component} = \begin{bmatrix} 1007.3 \\ 1177.9 \\ 1519.7 \\ 736.1 \\ 996.9 \end{bmatrix}}$$