Ma 309 — Matrix Algebra
Solutions for Practice Test

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1. Let A be the 4×4 matrix whose rows are v_1, v_2, v_3, v_4. The span of v_1, v_2, v_3, v_4 is preserved by elementary row operations on A, and the dimension of the span is the same as the row rank of A. Since

$$A = \begin{bmatrix} 1 & 3 & 4 & 0 \\ 0 & 5 & 2 & 0 \\ -1 & 7 & 0 & 0 \\ 2 & 1 & 1 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 4 & 0 \\ 0 & 5 & 2 & 0 \\ 0 & 10 & 4 & 0 \\ 0 & -5 & -7 & 5 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 3 & 4 & 0 \\ 0 & 5 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -5 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 4 & 0 \\ 0 & 1 & 2/5 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

the row rank and hence the dimension of $\mathrm{span}(v_1, v_2, v_3, v_4)$ is $\mathrm{rank}(A) = 3$. Since the row and column ranks of a matrix are the same, it would also be correct to use row operations on the matrix A^T with v_1, v_2, v_3, v_4 as columns.

2. The set of equations

$$\begin{bmatrix} 1 & 5 & 3 & 4 \\ 0 & 1 & 3 & 1 \\ 1 & 4 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad (*)$$

is preserved by elementary row operations on the matrix. The matrix in (*) is row equivalent to

$$\begin{bmatrix} 1 & 5 & 3 & 4 \\ 0 & 1 & 3 & 1 \\ 0 & -1 & -3 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -12 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus equation (*) is equivalent to the system

$$\begin{aligned} x_1 - 12x_3 - x_4 &= 0 \\ x_2 + 3x_3 + x_4 &= 0 \end{aligned} \qquad \text{or} \qquad \begin{aligned} x_1 &= 12x_3 + x_4 \\ x_2 &= -3x_3 - x_4 \end{aligned} \qquad (**)$$

If we add the two equations $x_3 = x_3$ and $x_4 = x_4$, the system (**) is equivalent to

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 12 \\ -3 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

This means that these two vectors span the set of solutions of (*). Since these two vectors are linearly independent, they are a basis of the solutions of (*).

3. The question is whether there exist constants c, d so that

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = c \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} + d \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad (*)$$

Looking just at the diagonal terms in (*), this implies $1 = 2c$ for the first diagonal term and $1 = c$ for the second diagonal term. Since c cannot be simultaneously $1/2$ and 1, the relation (*) cannot hold. Thus I_2 CANNOT be written as a linear combination of those two matrices.

4. The set of equations

$$\begin{bmatrix} 5 & 4 & 7 \\ 1 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$$

can be written in row equivalent form as

$$\left[\begin{array}{ccc|c} 5 & 4 & 7 & 4 \\ 1 & 2 & 1 & 2 \\ 2 & 1 & 3 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 0 & -6 & 2 & -6 \\ 1 & 2 & 1 & 2 \\ 0 & -3 & 1 & -3 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 2 \\ 0 & 1 & -1/3 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 0 & 5/3 & 0 \\ 0 & 1 & -1/3 & 1 \end{array}\right]$$

This is equivalent to the system

$$\begin{aligned} x_1 + (5/3)x_3 &= 0 \\ x_2 - (1/3)x_3 &= 1 \end{aligned} \qquad \text{or} \qquad \begin{aligned} x_1 &= -(5/3)x_3 \\ x_2 &= (1/3)x_3 + 1 \end{aligned} \qquad (**)$$

If we add the equation $x_3 = x_3$, this is equivalent to

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -5/3 \\ 1/3 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = c \begin{bmatrix} -5 \\ 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

for $c = x_3/3$.

Since the set of solutions DOES NOT include the zero vector $[0\ \ 0\ \ 0]^T$, it is NOT a vector subspace of R^3.

5. The condition that $x = [x_1\ \ x_2\ \ x_3\ \ x_4\ \ x_5]$ is orthogonal to v_1 and v_2 is equivalent to the system

$$\begin{bmatrix} 1 & 5 & 4 & 5 & 3 \\ 1 & 3 & 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad (*)$$

The system (*) is preserved by elementary row operations on the matrix, which is row equivalent to

$$\begin{bmatrix} 1 & 5 & 4 & 5 & 3 \\ 0 & -2 & -4 & -2 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 5 & 4 & 5 & 3 \\ 0 & 1 & 2 & 1 & 1 \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & 0 & -6 & 0 & -2 \\ 0 & 1 & 2 & 1 & 1 \end{bmatrix}$$

As in Problem 2, this is equivalent to the system

$$\begin{aligned} x_1 &= 6x_3 + 2x_5 \\ x_2 &= -2x_3 - x_4 - x_5 \end{aligned}$$

Adding the equations $x_3 = x_3$, $x_4 = x_4$, and $x_5 = x_5$, this is equivalent to

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} 6 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

It is easy to check that these three vectors are orthogonal to v_1 and v_2, and that the three vectors are a basis for $\{ v_1, v_2 \}^{\perp}$.

6. By definition, for $v = [2\ \ 1\ \ 0]$ and $u = [0\ \ 2\ \ 1]$, the projection of v onto u is that vector cu (that is, for some scalar c) such that cu and $v - cu$ are orthogonal. This is equivalent to $(cu, v - cu) = c(\,(u,v) - c(u,u)\,) = 0$ so that

$$c = \frac{(u,v)}{(u,u)} = \frac{2}{5}$$

Thus $cu = (2/5)[0\ \ 2\ \ 1] = [0\ \ 4/5\ \ 2/5]$ and $w = v - cu = [2\ \ 1/5\ \ -2/5]$. It is easy to check that

$$(cu, w) = \left(\begin{bmatrix} 0 \\ 4/5 \\ 2/5 \end{bmatrix}, \begin{bmatrix} 2 \\ 1/5 \\ -2/5 \end{bmatrix} \right) = 0$$

7. The characteristic polynomial of the matrix is

$$\det \begin{bmatrix} \lambda-2 & 2 & -3 \\ 0 & \lambda-3 & 2 \\ 0 & 1 & \lambda-2 \end{bmatrix} = (\lambda-2)\big((\lambda-3)(\lambda-2)-2\big)$$

$$= (\lambda-2)(\lambda^2-5\lambda+4) = (\lambda-2)(\lambda-1)(\lambda-4)$$

Since the eigenvalues are the roots of the characteristic polynomial, the eigenvalues are $\lambda_1 = 2$, $\lambda_2 = 1$, and $\lambda_3 = 4$.