Ma 309 — Matrix Algebra
Midterm Test #2

Prof. Sawyer — Washington Univ. — March 28, 2007

Calculators cannot be used. Closed textbook and notes.
Six problems on two pages.
Different parts of problems may not be equally weighted.

1. Find the rank of the matrix

$$A = \begin{bmatrix} 2 & 3 & 7 & 4 \\ 1 & 2 & 4 & 2 \\ -1 & 1 & 5 & 4 \end{bmatrix}$$

2. Let x, y be vectors in R^n (that is, $n \times 1$ column vectors).

(a) Find the dimensions of the matrices xy^T and x^Ty. (That is, $a \times b$ for what integers a and b?)

(b) Prove that $\mathrm{tr}(xy^T) = x^Ty$. (Recall that $\mathrm{tr}(A) = \sum_{i=1}^n a_{ii}$ if A is an $n \times n$ matrix with entries a_{ij}.)

3. Let $W = \{\, x \text{ in } R^4 : x_1 - x_4 = x_2 - x_3 \,\}$ where x_1, x_2, x_3, x_4 are the components of x.

(a) Show that W is a subspace of R^4. That is, if x, y are in R^4 and c is a real number, then cx in W and $x + y$ in W.

(b) Find a basis for the vector space W.

(c) What is the dimension of W? Why?

4. Let $w_1, w_2, \dots, w_n$ be orthogonal nonzero vectors in a vector space V with an inner product (u, v). Suppose that

$$x = \sum_{i=1}^n c_i w_i$$

where x in V for real numbers c_i. Find c_i in terms of x, w_i, and the inner product in V.

5. Define

$$(u, v)_A = \sum_{i=1}^n \sum_{j=1}^n A_{ij} u_i v_j$$

for u, v in R^n with components u_i, v_j for some $n \times n$ matrix A with entries $\{\, A_{ij} \,\}$. Show that a sufficient condition for

$$(Bu, v)_A = (u, Bv)_A \qquad \text{for all } u, v \text{ in } R^n$$

for an $n \times n$ matrix B is the relation $AB = B^TA$, or equivalently that $B^T = ABA^{-1}$ if A is invertible. (*Hint*: Expand all expressions into double or triple sums and see if you can find a pattern.)

6. Let $V = \text{span}\{ v_1, v_2 \}$ for $v_1 = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 3 \\ 1 \\ 7 \end{bmatrix}$ in R^3. Find a basis in R^3 for

$$V^\perp = \{ x \text{ in } R^3 : (x, v_1) = (x, v_2) = 0 \}$$

where (u, v) is the usual inner product (or dot product) in R^3.