Ma 309 — Matrix Algebra
Solutions for Midterm Test #2

Prof. Sawyer — Washington Univ. — March 28, 2007

1. Using elementary row operations:

$$A = \begin{bmatrix} 2 & 3 & 7 & 4 \\ 1 & 2 & 4 & 2 \\ -1 & 1 & 5 & 4 \end{bmatrix} \to \begin{bmatrix} 0 & 5 & 17 & 12 \\ 0 & 3 & 9 & 6 \\ -1 & 1 & 5 & 4 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & -1 & -5 & -4 \\ 0 & 1 & 3 & 2 \\ 0 & 5 & 17 & 12 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 & -2 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

after dividing the last row by 2. Since each row has a leading 1, $\text{rank}(A)=3$.

2. (a) Since x and y are $n \times 1$ column vectors, x^T and y^T are $1 \times n$ row vectors. Thus xy^T is $(n\times 1)\times(1\times n) = n\times n$ and $x^T y$ is $(1\times n)\times(n\times 1) = 1 \times 1$. In fact, $x^T y$ is the number $\sum_{i=1}^n x_i y_i$.

(b) By matrix multiplication,

$$xy^T = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} [y_1 \ \ y_2 \ \ \dots \ \ y_n] = \begin{bmatrix} x_1y_1 & x_1y_2 & \dots & x_1y_n \\ x_2y_1 & x_2y_2 & \dots & x_2y_n \\ \dots & \dots & \dots & \dots \\ x_ny_1 & x_ny_2 & \dots & x_ny_n \end{bmatrix}$$

If $A = xy^T$, then $A_{ij} = (xy^T)_{ij} = x_i y_j$. Thus $\text{tr}(A) = \text{tr}(xy^T) = \sum_{i=1}^n a_{ii} = \sum_{i=1}^n x_i y_i = x^T y$.

3. Here $W = \{\, x \text{ in } R^4 : x_1 - x_4 = x_2 - x_3 \,\}$ where x_1, x_2, x_3, x_4 are the components of x.

(a) Proof #1: The condition on $x = [x_1 \ \ x_2 \ \ x_3 \ \ x_4]^T$ to be in W is linear. One can then check that if x, y are in W, then the components of cx and $x + y$ satisfy the relation to be in W, so that $cx, x + y$ are in W as well. This means that W is a vector subpace of R^4.

Proof #2: If x satisfies the relation to be in W, then

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_2 - x_3 + x_4 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= x_2 v_1 + x_3 v_2 + x_4 v_3$$

where x_2, x_3, x_4 are arbitrary and v_1, v_2, v_3 are the three vectors on the right on the first line. Thus $W = \text{span}\{v_1, v_2, v_3\}$. Since the vector span of any set of vectors is automatically a vector space, W is a vector space.

(b) It is easy to check that the vectors v_1, v_2, v_3 above are linearly independent. Since they span W, they form a basis for W.

(c) Since W has a basis composed of 3 vectors, $\dim(W) = 3$.

4. Here $x = \sum_{i=1}^n c_i w_i = \sum_{j=1}^n c_j w_j$ where $w_1, w_2, \ldots, w_n$ are orthogonal nonzero vectors with respect to an inner product (u, v). This means that

$$(x, w_i) \;=\; \left(\sum_{j=1}^n c_j w_j,\, w_i\right) \;=\; \sum_{j=1}^n c_j\,(w_j,\, w_i) \;=\; c_i (w_i, w_i)$$

since $(w_i, w_j) = 0$ for $j \neq i$. Since $(w_i, w_i) > 0$ since w_i is nonzero, this implies $c_i = (x, w_i)/(w_i, w_i)$.

5. (This was the hardest question.)

Proof #1: If $(u, v)_0 = \sum_{i=1}^n u_i v_i$ is the usual inner product in R^n, then $(Cu, v)_0 = (u, C^T v)_0$ for any $n \times n$ matrix C. This is in the text: A short proof is

$$\begin{aligned}(Cu, v)_0 &= \sum_{i=1}^n (Cu)_i v_i = \sum_{i=1}^n \left(\sum_{k=1}^n C_{ik} u_k\right) v_i = \sum_{i=1}^n \sum_{k=1}^n C_{ik} u_k v_i \\ &= \sum_{k=1}^n u_k \sum_{i=1}^n C_{ik} v_i = \sum_{k=1}^n u_k \sum_{i=1}^n (C^T)_{ki} v_i = (u, C^T v)_0\end{aligned}$$

By definition, $(u, v)_A = (u, Av)_0$. Thus

$$\begin{aligned}(Bu, v)_A \;&=\; (Bu, Av)_0 \;=\; (Bu, (Av))_0 \;=\; (u, B^T Av)_0 \\ (u, Bv)_A \;&=\; (u, (Bv))_A \;=\; (u, A(Bv))_0 \;=\; (u, ABv)_0\end{aligned}$$

Hence if $AB = B^T A$, then $(Bu, v)_A = (u, Bv)_A$. (Note that a hint is not mandatory! There are other short proofs that do not involve expanding $(Bu, v)_A$ as a triple sum as in the second proof.)

Proof #2: Note

$$\begin{aligned}(Bu, v)_A \;&=\; \sum_{i=1}^n \sum_{j=1}^n A_{ij} (Bu)_i v_j \;=\; \sum_{i=1}^n \sum_{j=1}^n A_{ij} \sum_{k=1}^n B_{ik} u_k v_j \\ (u, Bv)_A \;&=\; \sum_{i=1}^n \sum_{j=1}^n A_{ij} u_i (Bv)_j \;=\; \sum_{i=1}^n \sum_{j=1}^n A_{ij} u_i \sum_{k=1}^n B_{jk} v_k\end{aligned}$$

The key is that you get two triple sums, not double sums. Note also

$$\sum_{i=1}^{n} A_{ij}B_{ik} = \sum_{i=1}^{n}(B^T)_{ki}A_{ij} = (B^TA)_{kj}$$
$$\sum_{j=1}^{n} A_{ij}B_{jk} = (AB)_{ik}$$

Thus by summing over i in the first triple sum and over j in the second triple sum,

$$\begin{aligned}(Bu,v)_A &= \sum_{j=1}^{n}\sum_{k=1}^{n}(B^TA)_{kj}u_kv_j &= \sum_{j=1}^{n}\sum_{i=1}^{n}(B^TA)_{ij}u_iv_j \\ (u,Bv)_A &= \sum_{i=1}^{n}\sum_{k=1}^{n}(AB)_{ik}u_iv_k &= \sum_{i=1}^{n}\sum_{j=1}^{n}(AB)_{ij}u_iv_j\end{aligned}$$

The two final sums are the same if $AB = B^TA$.

6. If $V = \text{span}\{v_1, v_2\}$, then x in $V^\perp$ if and only if $(x, v_1) = (x, v_2) = 0$ for $x = [x_1\ \ x_2\ \ x_3]^T$.

Proof #1: This is equivalent to two equations in three unknowns:

$$\begin{aligned} x_1 + 2x_2 + 4x_3 &= 0 \\ 3x_1 + x_2 + 7x_3 &= 0 \end{aligned} \tag{1}$$

By elementary row operations

$$\begin{bmatrix} 1 & 2 & 4 \\ 3 & 1 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 4 \\ 0 & -5 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

This means that the system (1) is equivalent to the system

$$\begin{aligned} x_1 + 2x_3 &= 0 \\ x_2 + x_3 &= 0 \end{aligned}$$

This in turn is equivalent to

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} = -x_3 \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

with no constraints on x_3. This means that $\{ [-2 \ \ -1 \ \ 1]^T \}$ is a basis for $V^\perp$.

Proof #2: A few people tried the following approach: (i) Use Gram-Schmidt orthogonalization to convert $\{ v_1, v_2 \}$ to an orthogonal basis $\{ v_1, w_2 \}$ for V where $w_2 = v_2 - cv_1$. Then (ii) use Gram-Schmidt orthogonalization again to extend $\{ v_1, w_2 \}$ to an orthogonal basis $\{ v_1, w_2, w_3 \}$ for all of R^3. Then one must have $V^\perp = \text{span}\{w_3\}$, so that $\{ w_3 \}$ is a basis for $V^\perp$.

This approach is OK, but is computationally more difficult. It is not difficult to find w_2 with integer entries, but no one was able to push through the argument to find w_3 (which by the first proof would have to be a constant times $[2 \ \ 1 \ \ -1]^T$).