Solutions to Section 2.7

2.59 (a) Let X be the number of occupied lanes, then X is binomial with $n = 10$ and $p = 0.75$. Then

$$P(X \leq 9) = 1 - P(X = 10) = 1 - \binom{10}{10}(0.75)^{10}(1-0.75)^0 = 0.944.$$

(b)

$$E(X) = np = 10 \times 0.75 = 7.5$$

and

$$\sigma_X = \sqrt{np(1-p)} = \sqrt{10 \times 0.75 \times (1 - 0.75)} = 1.369.$$

2.60 (a) Let X be the number of times a participant will be selected, then X is binomial with $n = 12$ and $p = 0.1$. Then

$$E(X) = np = 12 \times 0.1 = 1.2.$$

(b)

$$P(X = 2) = \binom{12}{2}(0.1)^2(1-0.1)^{10} = 0.230.$$

(c) Now $p = 0.2$, so that

$$P(X = 2) = \binom{12}{2}(0.2)^2(1-0.2)^{10} = 0.283.$$

2.61 (a)

$$P(X \leq 2) = \frac{\binom{10}{0}\binom{30}{4-0}}{\binom{40}{4}} + \frac{\binom{10}{1}\binom{30}{4-1}}{\binom{40}{4}} + \frac{\binom{10}{2}\binom{30}{4-2}}{\binom{40}{4}} = 0.300 + 0.444 + 0.214 = 0.958.$$

(b)

$$\begin{aligned} P(X \leq 2) &\approx \binom{4}{0}(0.25)^0(1-0.25)^4 + \binom{4}{1}(0.25)^1(1-0.25)^3 + \binom{4}{2}(0.25)^2(1-0.25)^2 \\ &= 0.316 + 0.422 + 0.211 = 0.949. \end{aligned}$$

The approximation is fairly accurate.

2.62 (a) X has a hypergeometric distribution with $N = 50$, $M = 2$, and $n = 3$. Then

$$f(x) = \frac{\binom{2}{x}\binom{48}{3-x}}{\binom{50}{3}} \text{ for } x = 0, 1, 2.$$

(b)

$$P(X=0) = \frac{\binom{2}{0}\binom{48}{3-0}}{\binom{50}{3}} = 0.882.$$

(c) Using $p = M/N = 2/50 = 0.04$,

$$P(X=0) \approx \binom{3}{0}(0.04)^0(1-0.04)^3 = 0.885.$$

The approximation is very accurate.

2.64 (a) X has a Binomial distribution with $n = 200$ and $p = 1/20 = 0.05$. Then

$$P(X \geq 5) = 1 - P(X \leq 4) = 1 - \sum_{i=0}^{4} \binom{200}{i}(0.05)^i(1-0.05)^{200-i}$$

(b) The Poisson distribution fits because an error on a page is a rare event, and the number of pages in the manuscript is large. Using $\lambda = 0.05 \times 200 = 10$, and the table of cumulative Poisson probabilities,

$$P(X \geq 5) = 1 - P(X \leq 4) = 1 - \sum_{i=0}^{4} \frac{e^{-10}10^i}{i!} = 1 - 0.029 = 0.971.$$

2.70 (a) Let E denote the event that the Eastern Conference team wins the series. Then

$$P(E) = P(E,\ 4 \text{ games}) + P(E,\ 5 \text{ games}) + P(E,\ 6 \text{ games}) + P(E,\ 7 \text{ games}).$$

For the Eastern Conference team to win in j games, it must have won 3 out of the first $j-1$ games, as well as the last one. So

$$P(E,\ j \text{ games}) = \binom{j-1}{3} p^4 q^{j-4}$$

and

$$P(E) = \binom{3}{3}p^4q^0 + \binom{4}{3}p^4q^1 + \binom{5}{3}p^4q^2 + \binom{6}{3}p^4q^3.$$

(b) Let W denote the event that the Western Conference team wins the series. Then the formula for the probability that the Western Conference team wins in j games is similar to that of the Eastern Conference, but with p and q switched. Then

$$\begin{aligned} P(\text{Series ends in } j \text{ games}) &= P(E,\ j \text{ games}) + P(W,\ j \text{ games}) \\ &= \binom{j-1}{3}p^4q^{j-4} + \binom{j-1}{3}q^4p^{j-4} \\ &= \binom{j-1}{3}\left[p^4q^{j-4} + q^4p^{j-4}\right]. \end{aligned}$$

Solutions to Section 2.8

2.71 (a)

$$P(90 \le X \le 100) = \frac{100-90}{100-70} = 0.333.$$

(b) To find the 90th percentile, set

$$P(b \le X \le 100) = \frac{100-b}{100-70} = 0.1.$$

Solving for b gives $b = 97$.

2.72 (a) If city A is at mile 0 and city C is at mile 75, then city B is at mile 25. Then the probability that the car is towed more than 10 miles is

$$P(10 \le X \le 15) + P(35 \le X \le 65) = \frac{5}{75} + \frac{30}{75} = 0.467.$$

(b) The probability that the car is towed more than 20 miles is

$$P(45 \le X \le 55) = \frac{10}{75} = 0.133.$$

(c)

$$\begin{aligned} P(\text{Towed} > 10 \text{ miles} | X > 20) &= \frac{P(\text{Towed} > 10 \text{ miles} \cap X > 20)}{P(X > 20)} \\ &= \frac{P(35 \le X \le 65)}{P(20 \le X \le 75)} \\ &= \frac{30/75}{55/75} = 0.545. \end{aligned}$$

2.73 (a) To find the pth quantile, set

$$F(x) = 1 - e^{-0.1x} = p$$

and solve for x, which yields

$$x = -10 \ln(1-p).$$

For $p = 0.5$, the median is 6.931. For $p = 0.75$, the 75th percentile is 13.863.

(b) There are no jobs in 15 minutes if the arrival time of the first job is above 15 minutes, so

$$P(\text{No jobs in 15 minutes}) = P(X > 15) = 1 - F(15) = e^{-0.1(15)} = 0.223.$$

2.74 (a) If the mean time to failure is 10,000 hours, then $\lambda = 1/10000$. To find the median, set

$$F(x) = 1 - e^{-x/10000} = 0.5$$

and solve for x, which yields

$$x = -10000 \ln(1 - 0.5) = 6931.472.$$

(b)

$$P(X \ge 1000) = 1 - F(1000) = e^{-1000/10000} = 0.905.$$

(c) Because of the memoryless property of the exponential distribution,

$$P(X \ge 2000 | X \ge 1000) = P(X \ge 1000) = 0.905.$$

2.75 Let X_i be the failure time of the ith bulb. Then X_i is exponential with failure rate $\lambda = 1/10000$. Since $T = X_1 + X_2 + \ldots + X_5$ is the sum of 5 exponential distributions with failure rate $\lambda = 1/10000$, T has a gamma distribution with the same failure rate $\lambda = 1/10000$ and $r = 5$. Then

$$E(T) = \frac{r}{\lambda} = \frac{5}{1/10000} = 50,000$$

and

$$\text{Var}(T) = \frac{r}{\lambda^2} = \frac{5}{(1/10000)^2} = 500,000,000.$$

2.76 Let X be the beta random variable under study. For $E(X)$ to be $3/4$,

$$\frac{a}{a+b} = \frac{3}{4}$$

or $a = 3b$. For $\text{Var}(X)$ to be $3/32$,

$$\frac{ab}{(a+b)^2(a+b+1)} = \frac{3b^2}{(4b)^2(4b+1)} = \frac{3}{32}$$

or

$$3 \times 32 = 3 \times 16 \times (4b+1)$$

or $b = 1/4$. Then solving for a yields $a = 3/4$.

Solutions to Section 2.9

2.78 (a)

$$P(Z \leq 1.68) = 0.9535,$$
$$P(Z > 0.75) = 1 - 0.7734 = 0.2266,$$
$$P(Z \leq -2.42) = 0.0078,$$
$$P(Z > -1) = 1 - 0.1587 = 0.8413.$$

(b)

$$P(1 \le Z \le 2) = 0.9772 - 0.8413 = 0.1359,$$
$$P(-2 \le Z \le -1) = 0.1359,$$
$$P(-1.5 \le Z \le 1.5) = 0.9332 - 0.0668 = 0.8664,$$
$$P(-1 \le Z \le 2) = 0.9772 - 0.1587 = 0.8185.$$

(c)

$$P(Z \le z_{.1}) = 1 - 0.1 = 0.9,$$
$$P(Z > -z_{.05}) = 1 - 0.05 = 0.95,$$
$$P(z_{.25} \le Z \le z_{.01}) = 0.25 - 0.01 = 0.24,$$
$$P(-z_{.25} \le Z \le z_{.01}) = 0.75 - 0.01 = 0.74.$$

2.79 (a)

$$z_{.3} = 0.525,\ z_{.15} = 1.04,\ \text{and } z_{.075} = 1.44.$$

(b) Since $x_p = \mu + z_p\sigma$,

$$x_{.3} = 4 + (0.525) \times 3 = 5.575,$$
$$x_{.15} = 4 + (1.04) \times 3 = 7.12,$$
$$x_{.075} = 4 + (1.44) \times 3 = 8.32.$$

2.80 (a) Let X be the weight of coffee in a can. Then

$$P(X < 16) = P\left(Z = \frac{X-\mu}{\sigma} < \frac{16-16.1}{0.5}\right) = P(Z < -0.2) = 0.4207.$$

(b)

$$\begin{aligned} P(16 < X < 16.5) &= P\left(\frac{16-16.1}{0.5} < Z = \frac{X-\mu}{\sigma} < \frac{16.5-16.1}{0.5}\right) \\ &= P(-0.2 < Z < 0.8) = 0.7881 - 0.4207 = 0.3674. \end{aligned}$$

(c) To find the 10th percentile, set

$$P\left(Z \le \frac{x_{.9}-\mu}{\sigma}\right) = 0.1$$

or

$$x_{.9} = \mu + z_{.9}\sigma = 16.1 + (-1.28) \times 0.5 = 15.46.$$

2.81 (a) W has a normal distribution.

(b)

$$E(W) = E(U) - E(V) = 160 - 120 = 40$$

and

$$\text{Var}(W) = \text{Var}(X) + \text{Var}(Y) = 30^2 + 25^2 = 1525.$$

(c)

$$P(U - V > 50) = P(W > 50) = P\left(Z = \frac{W - \mu}{\sigma} > \frac{50 - 40}{\sqrt{1525}}\right)$$
$$= P(Z > 0.256) = 1 - 0.6020 = 0.398.$$

2.82 (a) $X - Y$ has a normal distribution with

$$E(X - Y) = 0.526 - 0.525 = 0.001 \text{ and}$$

$$\sigma_{X-Y} = \sqrt{(3)^2 + (4)^2} = 5 \times 10^{-4}.$$

(b)

$$P(X-Y > 0) = P\left(Z = \frac{X - Y - \mu}{\sigma} > \frac{0 - 0.001}{5 \times 10^{-4}}\right) = P(Z > -2) = 1 - 0.0227 = 0.9773.$$

(c) Since the number of pairs that fit together has a binomial distribution with $n = 10$ and $p = 0.9772$, this probability is

$$\binom{10}{9}(0.9772)^9(0.0228)^1 + \binom{10}{10}(0.9772)^{10}(0.0228)^0 = 0.185 + 0.794 = 0.979.$$

2.83 (a) $\bar{X}$ has a normal distribution with mean $\mu = 90$ and SD $= \sigma/\sqrt{n} = 20/\sqrt{25} = 4$.

(b)

$$P(\bar{X} > 100) = P\left(Z = \frac{\bar{X} - \mu_{\bar{X}}}{\sigma_{\bar{X}}} > \frac{100 - 90}{4}\right) = P(Z > 2.5) = 0.0062.$$

(c) To find the 90th percentile, set

$$P\left(Z \le \frac{\bar{x}_{0.10} - \mu_{\bar{X}}}{\sigma_{\bar{X}}}\right) = 0.90$$

or

$$\bar{x}_{0.10} = \mu + z_{0.10}\sigma = 90 + (1.28) \times 4 = 95.12.$$