Answer Key

Ch. 2 # 35, 38, 40, 41, 42, 46

$\boxed{2.35}$ a. The pdf is $f(x) = \begin{cases} 1/N & \text{if } x = 1, 2, \ldots, N \\ 0 & \text{otherwise} \end{cases}$

b. The mean is

$$E(X) = \sum_x x f(x) = \sum_{x=1}^{N} x \times \frac{1}{N} = \frac{1}{N} \times \frac{N(N+1)}{2} = \frac{N+1}{2}$$

The variance is

$$Var(X) = \sum_x x^2 f(x) - \left(\frac{N+1}{2}\right)^2 = \frac{1}{N}\sum_{x=1}^{N} x^2 - \left(\frac{N+1}{2}\right)^2$$

$$= \frac{N(N+1)(2N+1)}{6N} - \left(\frac{N+1}{2}\right)^2 = \frac{N+1}{2}\left(\frac{2N+1}{3} - \frac{N+1}{2}\right)$$

$$= \frac{(N+1)(N-1)}{12}$$

c. For a single die, $N=6$, so that $E(X) = 7/2 = 3.5$

& $Var(X) = 7(5)/12 = 2.917$

$\boxed{2.38}$

$$E(X) = \sum_x x f(x) = \sum_{x=1}^{\infty} x \frac{1}{x(x+1)} = \sum_{x=1}^{\infty} \frac{1}{x+1} = \infty$$

$\boxed{2.40}$ a. The cdf is

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ \int_0^x \frac{1}{(1+z)^2}\,dz = \left.\frac{-1}{1+z}\right|_0^x = 1 - \frac{1}{1+x} & \text{if } x \geq 0 \end{cases}$$

b. To find the pth quantile, set

$$p = F(x) = 1 - \frac{1}{1+x}$$

& solve for x. In this case,

$$x = \frac{1}{1-p} - 1$$

For $p = 0.5$, $x_{(0.5)} = \frac{1}{1-0.5} - 1 = 1.$

ctd →

c. $E(\sqrt{X}) = \int_0^\infty \frac{\sqrt{x}}{(1+x)^2}\,dx$

Let $x = \tan^2\theta$, so that $dx = 2\tan\theta\sec^2\theta\,d\theta$. Then

$$E(\sqrt{X}) = \int_0^{\pi/2} \frac{2\tan^2\theta\sec^2\theta}{(\sec^2\theta)^2}\,d\theta = 2\int_0^{\pi/2} \sin^2\theta\,d\theta$$

$$= 2\frac{\pi}{4} = \frac{\pi}{2}$$

$\boxed{2.41}$ a. The cdf is

$$F(x) = \begin{cases} 0 & \text{if } x<1 \\ \int_1^x 2z^{-3}dz = -z^{-2}\Big|_1^x = 1-x^{-2} & \text{if } x \geq 1 \end{cases}$$

b. To find the pth quantile, set

$$p = F(x) = 1-x^{-2}$$

& solve for x. In this case,

$$x = \sqrt{\frac{1}{1-p}}$$

For $p=0.5$, $x_{(0.5)} = \sqrt{\frac{1}{1-0.5}} = \sqrt{2} = 1.414$

c. The mean is

$$E(X) = \int_1^\infty x\,(2x^{-3})\,dx = \int_1^\infty 2x^{-2}\,dx = -2x^{-1}\Big|_1^\infty = 2.$$

Since $E(X^2) = \int_1^\infty x^2(2x^{-3})\,dx = \int_1^\infty 2/x\,dx = 2\ln x\Big|_1^\infty = \infty$

the variance is also ∞.

$\boxed{2.42}$ a. For this to be a cdf,

$$\int_0^1 f(x) = \int_0^1 cx(1-x) = \left[cx^2/2 - cx^3/3\right]_0^1 = c/6 = 1, \quad \text{or } c=6.$$

b. The cdf is

$$F(x) = \begin{cases} 0 & \text{if } x<0 \\ \int_0^x 6z(1-z)\,dz = (3z^2-2z^3)\Big|_0^x = (3x^2-2x^3) & \text{if } 0\leq x<1 \\ 1 & \text{if } x\geq 1 \end{cases}$$

ctd →

c. The mean is

$$E(X) = \int_0^1 x(6x)(1-x)\,dx = \int_0^1 (6x^2 - 6x^3)\,dx = (2x^3 - 1.5x^4)\Big|_0^1 = 0.5$$

Since $E(X^2) = \int_0^1 x^2(6x)(1-x)\,dx = \int_0^1 (6x^3 - 6x^4)\,dx = (1.5x^4 - 1.2x^5)\Big|_0^1 = 0.3$,

the variance is $Var(X) = E(X^2) - E(X)^2 = 0.3 - 0.5^2 = 0.05$.

$\boxed{12.46}$ a. Let Y_i be Bernoulli(p). Then

$$\psi_{Y_i}(t) = E(e^{tY_i}) = \sum_{y=0}^{1} e^{ty} f(y) = pe^t + q.$$

Then

$$\psi_X(t) = \psi_{Y_1}(t) \cdots \psi_{Y_n}(t) = [\psi_{Y_i}(t)]^n = (pe^t + q)^n.$$

b. The first derivative of the moment generating function is

$$\psi'_X(t) = n(pe^t + q)^{n-1} pe^t.$$

Then $E(X) = \psi'_X(0) = n(p+q)^{n-1} p = np$.

The second derivative is

$$\psi''_X(t) = n(n-1)(pe^t + q)^{n-2} p^2 e^{2t} + n(pe^t + q)^{n-1} pe^t.$$

Then

$$E(X^2) = \psi''_X(0) = n(n-1)(p+q)^{n-2} p + n(p+q)^{n-1} p$$
$$= np[(n-1)p + 1],$$

and

$$Var(X) = E(X^2) - E(X)^2 = np[(n-1)p + 1] - (np)^2$$
$$= np(1-p).$$

Ch. 2 # 48, 49, 50, 52, 53, 54

$\boxed{2.48}$ a. $Cov(aX+b, cY+d) =$

$$E\left[\left(aX+b-(aE(x)+b)\right)\left(cY+d-(cE(Y)+d)\right)\right]$$

$$= acE\left[(X-E(X))(Y-E(Y))\right]$$

$$= ac\,Cov(X,Y)$$

b. $Var(X \pm Y) = E\left\{\left[(X \pm Y)-(E(X) \pm E(Y))\right]^2\right\}$

$$= E\left\{\left[(X-E(X)) \pm (Y-E(Y))\right]^2\right\}$$

$$= E\left[(X-E(X))^2\right] \pm 2E\left[(X-E(X))(Y-E(Y))\right] + E\left[(Y-E(Y))^2\right]$$

$$= Var(X) + Var(Y) \pm 2Cov(X,Y).$$

c. If X and Y are independent, then

$$E(XY) = \int_{x,y} xy\, f(x,y)\, dx\, dy$$

$$= \int_x \int_Y xy\, f(x) f(Y)\, dx\, dy$$

$$= \int_y y f(y)\, dy \int_{\cancel{X}} x f(x)\, dx = E(X)\, E(Y)$$

and

$$Cov(X,Y) = E(XY) - E(X)\,E(Y) = 0.$$

$\longrightarrow$

2.49 $\text{Cov}(X, Y) = \text{Cov}(X_1 + X_2, X_1 - X_2)$

$= E[(X_1 + X_2 - (E(X_1) + E(X_2))(X_1 - X_2 - (E(X_1) - E(X_2))]$

$= E[((X_1 - E(X_1)) + (X_2 - E(X_2)))((X_1 - E(X_1)) - (X_2 - E(X_2)))]$

$= E[(X_1 - E(X_1))^2 - (X_2 - E(X_2))^2]$

$= V(X_1) - V(X_2) = 0$

So the correlation ρ is also 0.

2.50 $\text{Cov}(Y_1, Y_2) = \text{Cov}(X_0 - X_1, X_0 - X_2)$

$= \text{Var}(X_0) - \text{Cov}(X_0, X_1) - \text{Cov}(X_0, X_2) + \text{Cov}(X_1, X_2)$

$= \sigma_0^2$

So the correlation is

$$\rho = \frac{\sigma_0^2}{\sqrt{\sigma_0^2 + \sigma_1^2}\sqrt{\sigma_0^2 + \sigma_2^2}}$$

2.52 a.

(x, y)	t	$f(x, y)$
(0,0)	0	0.3
(0,1)	1	0.18
(0,2)	2	0.12
(1,0)	1	0.15
(1,1)	2	0.09
(1,2)	3	0.06
(2,0)	2	0.05
(2,1)	3	0.03
(2,2)	4	0.02

→

b.

t	$f(t)$
0	0.3
1	0.33
2	0.26
3	0.09
4	0.02

c. $E(T) = \sum_t t f(t) = 0 \times 0.3 + 1 \times 0.33 + \ldots + 4 \times 0.02 = 1.2$

$$Var(T) = \sum_t t^2 f(t) - E(T)^2 = 0^2 \times 0.3 + 1^2 \times 0.33 + \ldots + 4^2 \times 0.02 - (1.2)^2 = 1.06$$

d. If heart and lung problems were positively correlated, E(T) would be slightly decreased. Since 0, 2, and 4 (when X=Y) would be more frequent and most of the probability is located at X=0, a positive correlation would increase the liklihood that Y was also low, and decrease the mean. The variance would be increased b/c, with the correlation between X and Y, extreme values like 0 or 4 would be more likely.

$\boxed{2.53}$ a. $E(X) = -200 \times 0.2 + 400 \times 0.8 = 280.$

$$\sigma_X = \sqrt{(-200-280)^2 \times 0.2 + (400-280)^2 \times 0.8} = 240$$

b. $E(Y) = -100 \times 0.1 + 300 \times 0.9 = 260$

$$\sigma_Y = \sqrt{(-100-260)^2 \times 0.1 + (300-260)^2 \times 0.9} = 120$$

→

c.

u	$f(u)$	v	$f(v)$
-300	0.02	-500	0.18
100	0.18	-100	0.02
300	0.08	100	0.72
700	0.72	500	0.08

d. $E(U) = -300 \times 0.02 + \ldots + 700 \times 0.72 = 540$

$\sigma_u = \sqrt{(-300)^2 \times 0.02 + \ldots + (700)^2 - (540)^2} = 268.328$

$E(V) = -500 \times 0.18 + \ldots + 300 \times 0.08 = 20$

$\sigma_v = \sqrt{(-500)^2 \times 0.18 + \ldots (500)^2 \times 0.08 - (20)^2} = 268.328$

e. $E(U) = 280 + 260 = 540$

$E(V) = 280 - 260 = 20$

$\sigma_u = \sigma_v = \sqrt{(240)^2 + (120)^2} = 268.328$

$\boxed{2.54}$ a.

x	$f(x)$	y	$f(y)$
100	0.5	0	0.25
250	0.5	100	0.25
		200	0.5

X and Y are not independent b/c

$P(X=100, Y=0) = 0.2 \neq P(X=100)\,P(Y=0) = 0.5 \times 0.25 = 0.125$

b.

y	$f(y \mid x=100)$	$f(y \mid x=250)$
0	0.4	0.1
100	0.2	0.3
200	0.4	0.6

→

c.

$$E(XY) = \sum_{x,y} xy f(x,y) = 100 \times 100 \times \frac{1500}{15000} + \dots + 250 \times 200 \times \frac{4500}{15000} = 23750,$$

$$E(X) = 175,$$

and

$$E(Y) = 0 \times 0.25 + 100 \times 0.25 + 200 \times 0.5 = 125.$$

Then

$$Cov(X,Y) = E(XY) - E(X)E(Y) = 23750 - (175)(125) = 1875.$$