Ma 439 — Linear Models — Fall 2010

Solutions for Problem Set #1 — Due September 16, 2010

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1. For the matrix

$$A = \begin{pmatrix} -2 & 9 & -7 & 3 & -2 \\ 5 & 13 & 14 & 6 & 0 \\ 11 & 0 & 17 & -2 & -3 \end{pmatrix}$$

(i) $a_{2+} = \sum_{j=1}^{5} a_{2j} = 5 + 13 + 14 + 6 + 0 = 38$
(ii) $\sum_{i=1}^{3} a_{i4} = 3 + 6 - 2 = 7$
(iii) $\sum_{i=1}^{3} a_{ii} = -2 + 13 + 17 = 28$
(iv) $\sum_{u=1}^{3} a_{u2}a_{u5} = 9 \times (-2) + 13 \times 0 + 0 \times (-3) = -18$

2. The 3×3 matrices

$$\text{(i)} \quad a_{ij} = i + j - 2 : \quad \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{pmatrix}$$

$$\text{(ii)} \quad a_{ij} = i^{j-1} : \quad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}$$

$$\text{(iii)} \quad a_{ij} = j - i : \quad \begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & 1 \\ -2 & -1 & 0 \end{pmatrix}$$

3. (i)

$$BB' = \begin{pmatrix} 1 & 7 \\ 4 & 3 \\ -3 & 6 \end{pmatrix} \begin{pmatrix} 1 & 4 & -3 \\ 7 & 3 & 6 \end{pmatrix} = \begin{pmatrix} 50 & 25 & 39 \\ 25 & 25 & 6 \\ 39 & 6 & 45 \end{pmatrix}$$

and

$$B'B = \begin{pmatrix} 1 & 4 & -3 \\ 7 & 3 & 6 \end{pmatrix} \begin{pmatrix} 1 & 7 \\ 4 & 3 \\ -3 & 6 \end{pmatrix} = \begin{pmatrix} 26 & 1 \\ 1 & 94 \end{pmatrix}$$

(ii) $\text{tr}(BB') = 50 + 25 + 45 = 120$ and $\text{tr}(B'B) = 26 + 94 = 120$

4.

$$A^2 = \begin{pmatrix} 1 & 3 & 7 \\ 2 & 6 & 14 \\ -1 & -3 & -7 \end{pmatrix} \begin{pmatrix} 1 & 3 & 7 \\ 2 & 6 & 14 \\ -1 & -3 & -7 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$A'A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 6 & -3 \\ 7 & 14 & -7 \end{pmatrix} \begin{pmatrix} 1 & 3 & 7 \\ 2 & 6 & 14 \\ -1 & -3 & -7 \end{pmatrix} = \begin{pmatrix} 6 & 18 & 42 \\ 18 & 54 & 126 \\ 42 & 126 & 294 \end{pmatrix}$$

The matrices are not the same. (That is, $A^2 \neq A'A$.)

5. (i)

$$j'A = (\,1 \quad \ldots \quad 1\,) \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \ddots & \ddots & \ddots \\ a_{n1} & \ldots & a_{nn} \end{pmatrix} = (\,a_{+1} \quad \ldots \quad a_{+n}\,)$$

where a_{+j} means the sum over the j^{th} column, and

(ii)

$$Aj = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \ddots & \ddots & \ddots \\ a_{n1} & \ldots & a_{nn} \end{pmatrix} \begin{pmatrix} 1 \\ \ldots \\ 1 \end{pmatrix} = \begin{pmatrix} a_{1+} \\ \ldots \\ a_{n+} \end{pmatrix}$$

where a_{i+} means the sum over the i^{th} row.

6. (i) $A^2 = (xy')(xy') = x(y'x)y' = (y'x)xy' = (y'x)A = A$

(ii) Similarly, $A^2 = (y'x)A = 0$

(iii) $\text{tr}(A) = \text{tr}(\{x_i y_j\}) = \sum_{i=1}^{3} x_i y_i = x'y$.

Alternatively, since $\text{tr}(AB) = \text{tr}(BA)$ for any pair of matrices A, B for which both AB and BA are defined, you can also argue $\text{tr}(xy') = \text{tr}(y'x) = y'x$ since the trace of a 1×1 matrix is just its value.

(iv) The range of A is $\{\, Az : z \in R^3 \,\} = \{\, xy'z : z \in R^3 \,\} = \{\, x(y'z) : z \in R^3 \,\} = \{\, \lambda x : \lambda \in R^1 \,\}$ since $y \neq 0$ by assumption. Since $x \neq 0$ by assumption also, this is the line generated by x where $x \neq 0$. Since this line is one-dimensional, range$(A) = 1$.

7. (i) For fixed i and j, $(AA')_{ij} = \sum_{k=1}^{n} a_{ik}a_{jk}$ is the dot product of the i^{th} and j^{th} rows of A, but also, for fixed k, $a_{ik}a_{jk} = (u_k u_k')_{ij}$ is the outer product of the k^{th} column u_k with itself. Recall that, for two column vectors x, y, the outer product xy' is defined by $(xy')_{ij} = x_i y_j$. Thus $AA' = \sum_{k=1}^{n} u_k u_k'$.

(ii) For fixed i and j, $(A'A)_{ij} = \sum_{k=1}^{n} a_{ki} a_{kj}$ is the dot product of the i^{th} and j^{th} column of A, but, for fixed k, $a_{ki} a_{kj} = (b_k b_k')_{ij}$ is the outer product of k^{th} row r_k viewed as a column vector $b_k = r_k'$ with itself. Thus $A'A = \sum_{k=1}^{n} b_k b_k'$.