Ma 439 — Linear Models — Fall 2010

Solutions for Problem Set #2 — Due October 19, 2010

Prof. Sawyer — Washington University

(Do problems 1-4 by hand, and problems 5-6 using SAS.)

1. (10) Since $\mathrm{Cov}(X,Y)$ for real-valued random variables X,Y is linear in both variables and $\mathrm{Cov}(X_2,X_2) = \mathrm{Cov}(X_2,X_1)$,

$$\begin{aligned}\mathrm{Cov}(X_1 - X_2, X_1 + X_2) &= \mathrm{Cov}(X_1, X_1 + X_2) - \mathrm{Cov}(X_1, X_1 + X_2)\\ &= \mathrm{Cov}(X_1,X_1) + \mathrm{Cov}(X_1,X_2) - \mathrm{Cov}(X_2,X_1) - \mathrm{Cov}(X_2,X_2)\\ &= \mathrm{Cov}(X_1,X_1) - \mathrm{Cov}(X_2,X_2) = \mathrm{Var}(X_1) - \mathrm{Var}(X_2)\end{aligned}$$

2. (10) Since $A = A'$, $A = RDR'$ where R is an orthogonal matrix and $D = \mathrm{diag}(\lambda_1, \lambda_2)$ by the spectral theorem. Then $\mathrm{tr}(A) = \mathrm{tr}\big((RD)R'\big) = \mathrm{tr}\big(R'(RD)\big) = \mathrm{tr}(RR'D) = \mathrm{tr}(D) = \lambda_1 + \lambda_2 > 0$ and $\det(A) = \det(RDR') = \det(R)\det(D)\det(R') = \det(D)\det(R)^2 = \lambda_1\lambda_2 > 0$. Since $\lambda_1\lambda_2 > 0$, either (i) $\lambda_1 > 0$ and $\lambda_2 > 0$ or (ii) $\lambda_1 < 0$ and $\lambda_2 < 0$. The condition $\lambda_1 + \lambda_2 > 0$ eliminates case (ii), so that both $\lambda_1 > 0$ and $\lambda_2 > 0$. This implies that A is positive definite. (There are several ways to do this problem.)

3. (20) The random column vector $X = (X_1\ X_2\ X_3)'$ has $E(X) = 0$ and covariance matrix

$$A = \mathrm{Cov}(X) = \begin{pmatrix} 3 & -4 & 1 \\ -4 & 10 & -2 \\ 1 & -2 & 3 \end{pmatrix}$$

Let $Y = X_1 + 2X_2 + 3X_3$ and $Z = X_2 + 4X_3$.

(i) If $A = \mathrm{Cov}(X)$, then $\mathrm{Var}(X_i) = A_{ii}$. Reading entries from the diagonal of A, $\mathrm{Var}(X_1) = A_{11} = 3$ and $\mathrm{Var}(X_2) = A_{22} = 10$.

(ii) Since $Y = v'X$ for $v = (1\ 2\ 3)'$ and $Z = w'X$ for $w = (0\ 1\ 4)'$

$$\mathrm{Var}(Y) = \mathrm{Var}(v'X) = v'\,\mathrm{Cov}(X)v = (\,1 \quad 2 \quad 3\,)\begin{pmatrix} 3 & -4 & 1 \\ -4 & 10 & -2 \\ 1 & -2 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$= (\,1 \quad 2 \quad 3\,)\begin{pmatrix} -2 \\ 10 \\ 6 \end{pmatrix} = 36$$

$$\mathrm{Var}(Z) = \mathrm{Var}(w'X) = w'\,\mathrm{Cov}(X)w = (\,0 \quad 1 \quad 4\,)\begin{pmatrix} 3 & -4 & 1 \\ -4 & 10 & -2 \\ 1 & -2 & 3 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 4 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 10 \end{pmatrix} = 42$$

(iii)

$$\begin{aligned} \mathrm{Cov}(Y, Z) &= \mathrm{Cov}\left(\sum_{i=1}^{3} v_i X_i, \sum_{j=1}^{3} w_j X_j \right) = \sum_{i=1}^{3} \sum_{j=1}^{3} v_i w_j \,\mathrm{Cov}(X_i, X_j) \\ &= v' \,\mathrm{Cov}(X) w = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 3 & -4 & 1 \\ -4 & 10 & -2 \\ 1 & -2 & 3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 10 \end{pmatrix} = 34 \end{aligned}$$

(iv)

$$\mathrm{Cov}(W) = \begin{pmatrix} \mathrm{Cov}(X_1, X_1) & \mathrm{Cov}(X_1, X_3) \\ \mathrm{Cov}(X_3, X_1) & \mathrm{Cov}(X_3, X_3) \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

4. (20) (i) If $Y = AX$ where A is 2×3 and X is 3×1 (that is, $X \in R^3$), then Y is 2×1 (that is, $Y \in R^2$).
(ii) $\mu_Y = E(Y) = AE(X) = A\mu_X$ and $C = \mathrm{Cov}(Y) = A\,\mathrm{Cov}(X)A'$, so that

$$\mu_Y = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 5 \\ -3 \\ -2 \end{pmatrix} = \begin{pmatrix} -7 \\ -7 \end{pmatrix}$$

$$\begin{aligned} C &= \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 3 & 5 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 13 & 8 \\ 21 & 13 \end{pmatrix} \\ &= \begin{pmatrix} 89 & 55 \\ 55 & 34 \end{pmatrix} \end{aligned}$$

5. (20) (i) See Page 1 in `HW2.lst` (The SAS source file is `HW2.sas`.)

(ii) By definition (see for example Section 9 page 18 in the Multivariate Linear Models handout on the Math 439 Web site), $Q \approx W(6, 40, I_6)$ if we can write $Q = \sum_{i=1}^{40} Z_i Z_i'$ where $Z_i = (Z_{i1}, \ldots, Z_{i6})'$ are independent $N(0, I_6)$, or equivalently if $Q_{ab} = \sum_{i=1}^{40} Z_{ia} Z_{ib}$ where $\{ Z_{ia} \}$ are independent real-valued $N(0, 1)$ for $1 \leq a, b \leq 6$ and $1 \leq i \leq 40$. This is exactly how W was constructed.

(iii) See the bottom of Page 1 and the top of Page 2 in `HW2.lst`

(iv,v,vi) See Page 2 in `HW2.lst`

6. (20) (i) We reject H_0 with $P < 0.0001$. (See Page 5 in `HW2.lst`.)

(ii) $F = 30.67$ with $d = 4$ degrees of freedom in the numerator and $m + n - 2 - d + 1 = 19 + 20 - 2 - 4 + 1 = 34$ degrees of freedom in the denominator.

(iii) All four measurements y_1, y_2, y_3, y_4 are highly significantly different between the two beetle species. The P-values were

Variable	EqualVariance	Satterthwaite
y_1	0.0004	0.0005
y_2	0.0004	0.0004
y_3	< 0.0001	< 0.0001
y_3	< 0.0001	< 0.0001