Ma 494 — Theoretical Statistics

Solutions for Test #1 — February 19, 2009

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Closed book and closed notes. One $8\frac{1}{2} \times 11$ sheet of paper with notes on both sides and a calculator are allowed.

1. The likelihood for $X = (X_1, \dots, X_n)$ is

$$L(r, X_1, \dots, X_n) \;=\; \prod_{j=1}^{n} r \exp(-r/X_j)(1/X_j^2) \;=\; r^n \prod_{j=1}^{n} \exp\left(-\frac{r}{X_j}\right) \prod_{j=1}^{n} \frac{1}{X_j^2}$$

$$= r^n \exp\left(-\sum_{j=1}^{n} \frac{r}{X_j}\right) \prod_{j=1}^{n} \frac{1}{X_j^2}$$

Thus

$$\log L(r, X) = n \log r - r \sum_{j=1}^{n} \frac{1}{X_j} \qquad \text{and} \qquad \frac{d}{dr} \log L(r, X) = \frac{n}{r} - \sum_{j=1}^{n} \frac{1}{X_j}$$

Setting $\frac{d}{dr} \log L(r, X) = 0$ gives

$$\frac{1}{r} = \frac{1}{n} \sum_{j=1}^{n} \frac{1}{X_j} \qquad \text{and} \qquad \widehat{r} = \frac{n}{\sum_{j=1}^{n} \frac{1}{X_j}}$$

for the maximum-likelihood estimator.

2. If $X_1, \dots, X_n$ are a sample from $f(x, \theta)$, then $E(\overline{X}) = (1/n) \sum_{j=1}^{n} E(X_j) = E(X_1)$ where $E(X_1) = \int_{-\infty}^{\infty} x f(x, \theta)\, dx$, so that it is sufficient to prove $E(X_1) = \theta$. The easiest way to calculate the integral is to write it as

$$\int_{-\infty}^{\infty} x(1/2) \exp(-|x - \theta|)\, dx \;=\; \int_{-\infty}^{\infty} (x + \theta)(1/2) \exp(-|x|)\, dx$$

$$= \int_{-\infty}^{\infty} x(1/2) \exp(-|x|)\, dx \;+\; \int_{-\infty}^{\infty} \theta(1/2) \exp(-|x|)\, dx$$

The first integral after the last equals sign is zero since $\exp(-|x|)$ is an even function. The second integral is θ since the integral of a probability density is one. Thus $E(X_i) = \theta$, which was to be proven.

(You can also calculate the first integral by breaking it up into two pieces, one for $x \le \theta$ and one for $x \ge \theta$.)

3. (i) If $f(x,\mu) = e^{-\mu}\mu^x/x!$, then

$$\log f(x,\mu) = -\mu + x\log\mu - \log(x!) \qquad \text{and} \qquad \frac{d}{d\mu}\log f(x,\mu) = -1 + \frac{x}{\mu}$$

Thus the scores are

$$Y_k \;=\; \frac{d}{d\mu}\log f(X_k,\mu) \;=\; -1 + \frac{X_k}{\mu} \;=\; \frac{X_k - \mu}{\mu}$$

The Fisher information can be calculated either as

$$I(\mu) \;=\; \mathrm{Var}(Y_k) \;=\; \frac{\mathrm{Var}(X_k)}{\mu^2} \;=\; \frac{1}{\mu}$$

or as

$$I(\mu \;=\; -E\left(\frac{d^2}{d\mu^2}\log f(x,\mu)\right) \;=\; -E\left(-\frac{X_k}{\mu^2}\right) \;=\; \frac{1}{\mu}$$

since $E(X_k) = \mathrm{Var}(X_k) = \mu$.

(ii) The Cramér-Rao lower bound for the variance of an unbiased estimator is $1/\Big(nI(\mu)\Big) = \mu/n$ in this case. Since $\mathrm{Var}(\overline{X}) = \mathrm{Var}(X_k)/n = \mu/n$ as well, $\overline{X}$ is an efficient estimator of μ.

4. If $f(x,p) = (x+1)p^x(1-p)^2$ for $x = 0,1,2,\ldots$, the likelihood is

$$L(p, X_1,\ldots,X_n) \;=\; \prod_{j=1}^{n}(X_j+1)p^{X_j}(1-p)2 \;=\; (1-p)^{2n}p^{\left(\sum_{j=1}^{n} X_j\right)}\prod_{j=1}^{n}(X_j+1)$$

Thus if $X = (X_1,\ldots,X_n)$

$$\log L(p,X) \;=\; 2n\log(1-p) \;+\; \sum_{j=1}^{n} X_j\log(p) \;+\; A(X)$$

and

$$\frac{d}{dp}\log L(p,X) \;=\; \frac{-2n}{1-p} \;+\; \left(\sum_{j=1}^{n} X_j\right)\frac{1}{p}$$

(notice the minus sign from differentiating $\log(1-p)$). Setting $\frac{d}{dp}\log L(p,X) = 0$ gives $p/(1-p) = (1/2n)\sum_{j=1}^{n} X_j = (1/2)\overline{X}$ and $p = \widehat{p} = \overline{X}/(2+\overline{X})$.

5. If $f(x, \theta) = 4\theta x^3 e^{-\theta x^4}$, the likelihood is

$$\begin{aligned} L(\theta, X_1, \ldots, X_n) &= \prod_{j=1}^{n} 4\theta X_j^3 e^{-\theta X_j^4} = (4\theta)^n \exp\left(-\theta \sum_{j=1}^{n} X_j^4\right) \prod_{j=1}^{n} X_j^3 \\ &= g\left(\theta, \sum_{j=1}^{n} X_j^4\right) A(X_1, \ldots, X_n) \end{aligned}$$

for $g(\theta, y) = (4\theta)^n e^{-\theta y}$ and $A(X) = \prod_{j=1}^{n} X_j^3$. This implies that $S(X) = \sum_{j=1}^{n} X_j^4$ is a sufficient statistic for θ.