Ma 5051 — Real Variables and Functional Analysis

Solutions for Problem Set #1 due September 10, 2009

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See `m5051hw1.tex` for problem text.

1. Since the reverse set inclusion follows from $\mathcal{C}_0 \subseteq \mathcal{E}$ implies $\mathcal{M}(\mathcal{C}_0) \subseteq \mathcal{M}(\mathcal{E})$, it is sufficient to prove

$$\mathcal{M}(\mathcal{E}) \subseteq (\text{Set union})\{\, \mathcal{M}(\mathcal{C}_0) : \text{countable } \mathcal{C}_0 \subseteq \mathcal{E}\,\} \tag{1}$$

Let $\mathcal{M}_1$ be the right-hand side of (1). If we can show that $\mathcal{E} \subseteq \mathcal{M}_1$ and that $\mathcal{M}_1$ is a σ-algebra, then $\mathcal{E} \subseteq \mathcal{M}(\mathcal{E}) \subseteq \mathcal{M}_1$ from the definition of $\mathcal{M}(\mathcal{E})$ and (1) would follow.

Since any $E \in \mathcal{E}$ implies $E \in \{\, E\,\} \subseteq \mathcal{M}(\{\, E\,\})$, we conclude $\mathcal{E} \subseteq \mathcal{M}_1$. To show that $\mathcal{M}_1$ is a σ-algebra, we need to show (i) $\phi \in \mathcal{M}_1$, (ii) $A \in \mathcal{M}_1$ implies $A^c \in \mathcal{M}_1$, and (iii) $\{\, A_i\,\} \subseteq \mathcal{M}_1$ implies $\cup_{i=1}^{\infty} \in \mathcal{M}_1$. For (i), $\phi \in \mathcal{M}(\mathcal{C}_0)$ for any $\mathcal{C}_0$ even if $\mathcal{C}_0 = \phi$, so $\phi \in \mathcal{M}_1$. For (ii), if $A \in \mathcal{M}_1$, then $A \in \mathcal{M}(\mathcal{C}_0)$ for some countable $\mathcal{C}_0 \subseteq \mathcal{E}$, so $A^c \in \mathcal{M}(\mathcal{C}_0) \subseteq \mathcal{M}_1$. For (iii), assume $A_i \in \mathcal{M}(C_i)$ for countable sets $\mathcal{C}_i \subseteq \mathcal{E}$. Then $C = \cup_{i=1}^{\infty} C_i \subset \mathcal{E}$ is also countable, and $A_i \in \mathcal{M}(\mathcal{C})$ for all i. Thus $\cup_{i=1}^{\infty} A_i \in \mathcal{M}(\mathcal{C}) \subseteq \mathcal{M}_1$, which show that $\mathcal{M}_1$ is a σ-algebra, which completes the proof of the result.

2. Let $\mathcal{E} = \{\, S_r(x) : r > 0,\, x \in X\,\}$. Since $\mathcal{E} \subseteq \tau$, $\mathcal{M}(\mathcal{E}) \subseteq \mathcal{M}(\tau)$. I now claim $\tau \subseteq \mathcal{M}(\mathcal{E})$, which would prove $\mathcal{M}(\tau) \subseteq \mathcal{M}(\mathcal{E})$ and hence $\mathcal{M}(\tau) = \mathcal{M}(\mathcal{E})$. Let $\{\, y_n : n = 1, 2, \ldots\,\} \subseteq X$ be a sequence with $\overline{\{\, y_n\,\}} = X$. Let $\mathcal{O} \in \tau$ be an arbitrary open set. If $y_m \in \mathcal{O}$, let $r_m = d(y_m, \partial\mathcal{O}) = \sup r : S_r(y_m) \subseteq \mathcal{O}$. Then $r_m > 0$ and $S_{r_m}(y_m) \subseteq \mathcal{O}$. For a general $y \in \mathcal{O}$, there exist $y_{m_i} \in \mathcal{O}$ such that $y_{m_i} \to y$ as $i \to \infty$. Then $r = d(y, \partial\mathcal{O}) > 0$ and $\liminf_{i\to\infty} r_{y_i} \ge r$. It follows that $y \in S_{y_{m_i}}$ for sufficiently large i and hence

$$\mathcal{O} = \cup S_{r_m}(y_m) : y_m \in \mathcal{O} \tag{2}$$

since the right-hand side of (2) is countable, it follows that $\mathcal{O} \in \mathcal{M}(\mathcal{E})$ and hence $\mathcal{M}(\tau) = \mathcal{M}(\mathcal{E})$. (One could also argue from the fact that a separable metric space is second countable.)

3. First, given $E, F, G \in \mathcal{M}$, note that $E - G \subseteq (E - F) \cup (F - G)$. The same set inequality with E, F, G replaced by G, F, E implies

$$\mu(EMG) \le \mu(EMF) + \mu(FMG) \tag{3}$$

. where $EMG = (E - G) \cup (G - E)$.

(a) Note $E \subseteq F \cup (E - F)$ and $F \subseteq E \cup (F - E)$. If $\mu(EMF) = 0$, then $\mu(E) \le \mu(F) \le \mu(E)$ and $\mu(E) = \mu(F)$.

(b) For $\sim$ to be an equivalence relation of $\mathcal{M}$, we need that for all $E, F, G \in \mathcal{M}$, (i) $E \sim E$ (for all $E \in \mathcal{M}$), (ii) $E \sim F$ implies $F \sim E$, and (iii) if $E \sim F$ and $F \sim G$, then $E \sim G$. Since $E \sim F$ means $\mu(EMF) = 0$ and $EMF = FME$, (i) and (ii) follow, and (iii) follows from (3).

(c) Define $\rho(E, F) = \mu(E \triangle F)$. Then $\rho(E, G) \leq \rho(E, F) + \rho(F, G)$ by (3) and ρ defines a metric on the space of equivalence classes $\mathcal{M}/\sim$ by basic properties of equivalence classes.

4. Set $\mu^*(E) = \sqrt{\mathrm{card}(E)}$ for $E \subseteq X$.

(a) We must show that (i) $\mu^*(\phi) = 0$, (ii) $E \subseteq F$ implies $\mu^*(E) \leq \mu^*(F)$, and (iii) $\mu^*(\cup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} \mu^*(E_i)$ for arbitrary sets $E, F, E_i \subseteq X$. Properties (i) and (ii) follow from the fact that $\mu^*(E)$ is a monotonic function of the number of elements in E. For (iii),

$$\begin{aligned} \mu^*\Big(\bigcup_{i=1}^{\infty} E_i\Big) &= \sqrt{\mathrm{card}\Big(\bigcup_{i=1}^{\infty} E_i\Big)} \leq \sqrt{\sum_{i=1}^{\infty} \mathrm{card}(E_i)} \\ &= \sqrt{\sum_{i=1}^{\infty} x_i} \leq \sum_{i=1}^{\infty} \sqrt{x_i}, \qquad x_i = \sqrt{\mathrm{card}(E_i)} \end{aligned}$$

follows from the inequality

$$\sum_{i=1}^{\infty} x_i \leq \Big(\sum_{i=1}^{\infty} \sqrt{x_i}\Big)^2 = \sum_{i=1}^{\infty} x_i + \sum\sum_{i<j} \sqrt{x_i x_j}$$

(b) Assume $x \in A$ and $y \notin A$ for $A \in \mathcal{F}(\mu^*)$ and set $E = \{x, y\}$. Then the condition for $A \in \mathcal{F}(\mu^*)$ is

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c) \tag{4}$$

for all subsets $E \subseteq X$. If $x \in A$ and $y \notin A$ and $E = \{x, y\}$, then $\mu^*(E) = \sqrt{2}$ and $\mu^*(E \cap A) = \mu^*(E \cap A^c) = 1$, which contradicts (5). Thus $A \in \mathcal{F}(\mu^*)$ can only occur if $A = \phi$ or $A = X$, so that $\mathcal{F}(\mu^*) = \{\phi, X\}$.