Ma 5051 — Real Variables and Functional Analysis
Solutions for Problem Set #1 due September 10, 2009
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See m5051hw1.tex for problem text.

1. Since the reverse set inclusion follows from $C_0 \subseteq \mathcal{E}$ implies $M(C_0) \subseteq M(\mathcal{E})$, it is sufficient to prove

$$M(\mathcal{E}) \subseteq \{ \text{Set union}\{ M(C_0) : \text{countable } C_0 \subseteq \mathcal{E}\}$$

Let $M_1$ be the right-hand side of (1). If we can show that $\mathcal{E} \subseteq M_1$ and that $M_1$ is a $\sigma$-algebra, then $\mathcal{E} \subseteq M(\mathcal{E}) \subseteq M_1$ from the definition of $M(\mathcal{E})$ and (1) would follow.

Since any $E \in \mathcal{E}$ implies $E \in \{ E \} \subseteq M(\{ E \})$, we conclude $\mathcal{E} \subseteq M_1$. To show that $M_1$ is a $\sigma$-algebra, we need to show (i) $\phi \in M_1$, (ii) $A \in M_1$ implies $A^c \in M_1$, and (iii) $\{ A_i \} \subseteq M_1$ implies $\bigcup_{i=1}^{\infty} A_i \in M_1$. For (i), $\phi \in M(C_0)$ for any $C_0$ even if $C_0 = \phi$, so $\phi \in M_1$. For (ii), if $A \in M_1$, then $A \in M(C_0)$ for some countable $C_0 \subseteq \mathcal{E}$, so $A^c \in M(C_0) \subseteq M_1$. For (iii), assume $A_i \in M(C_i)$ for countable sets $C_i \subseteq \mathcal{E}$. Then $C = \bigcup_{i=1}^{\infty} C_i \subseteq \mathcal{E}$ is also countable, and $A_i \in M(C)$ for all $i$. Thus $\bigcup_{i=1}^{\infty} A_i \in M(C) \subseteq M_1$, which show that $M_1$ is a $\sigma$-algebra, which completes the proof of the result.

2. Let $\mathcal{E} = \{ S_{r}(x) : r > 0, x \in X \}$. Since $\mathcal{E} \subseteq \tau$, $M(\mathcal{E}) \subseteq M(\tau)$. I now claim $\tau \subseteq M(\mathcal{E})$, which would prove $M(\tau) \subseteq M(\mathcal{E})$ and hence $M(\tau) = M(\mathcal{E})$. Let $\{ y_n : n = 1, 2, \ldots \} \subseteq X$ be a sequence with $\{ y_n \} = X$. Let $O \in \tau$ be an arbitrary open set. If $y_m \in O$, let $r_m = d(y_m, \partial O) = \sup r : S_r(y_m) \subseteq O$. Then $r_m > 0$ and $S_{r_m}(y_m) \subseteq O$. For a general $y \in O$, there exist $y_{m_i} \in O$ such that $y_{m_i} \rightarrow y$ as $i \rightarrow \infty$. Then $r = d(y, \partial O) > 0$ and $\liminf_{i \rightarrow \infty} r y_i \geq r$. It follows that $y \in S_{y_{m_i}}$ for sufficiently large $i$ and hence

$$O = \bigcup S_{r_m}(y_m) : y_m \in O$$

since the right-hand side of (2) is countable, it follows that $O \in M(\mathcal{E})$ and hence $M(\tau) = M(\mathcal{E})$. (One could also argue from the fact that a separable metric space is second countable.)

3. First, given $E, F, G \in M$, note that $E - G \subseteq (E - F) \cup (F - G)$. The same set inequality with $E, F, G$ replaced by $G, F, E$ implies

$$\mu(EMG) \leq \mu(EMF) + \mu(FMG)$$

where $EMG = (E - G) \cup (G - E)$.

(a) Note $E \subseteq F \cup (E - F)$ and $F \subseteq E \cup (F - E)$. If $\mu(EMF) = 0$, then $\mu(E) \leq \mu(F) \leq \mu(E)$ and $\mu(E) = \mu(F)$. 

(b) For $\sim$ to be an equivalence relation of $\mathcal{M}$, we need that for all $E, F, G \in \mathcal{M}$, (i) $E \sim E$ (for all $E \in \mathcal{M}$), (ii) $E \sim F$ implies $F \sim E$, and (iii) if $E \sim F$ and $F \sim G$, then $E \sim G$. Since $E \sim F$ means $\mu(EMF) = 0$ and $EMF = FME$, (i) and (ii) follow, and (iii) follows from (3).

(c) Define $\rho(E, F) = \mu(E \Delta F)$. Then $\rho(E, G) \leq \rho(E, F) + \rho(F, G)$ by (3) and $\rho$ defines a metric on the space of equivalence classes $\mathcal{M}/\sim$ by basic properties of equivalence classes.

4. Set $\mu^*(E) = \sqrt{\text{card}(E)}$ for $E \subseteq X$.

(a) We must show that (i) $\mu^*(\emptyset) = 0$, (ii) $E \subseteq F$ implies $\mu^*(E) \leq \mu^*(F)$, and (iii) $\mu^*(\bigcup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} \mu^*(E_i)$ for arbitrary sets $E, F, E_i \subseteq X$. Properties (i) and (ii) follow from the fact that $\mu^*(E)$ is a monotonic function of the number of elements in $E$. For (iii),

$$\mu^* \left( \bigcup_{i=1}^{\infty} E_i \right) = \sqrt{\text{card} \left( \bigcup_{i=1}^{\infty} E_i \right)} \leq \sqrt{\sum_{i=1}^{\infty} \text{card}(E_i)}$$

$$= \sqrt{\sum_{i=1}^{\infty} x_i} \leq \sum_{i=1}^{\infty} \sqrt{x_i}, \quad x_i = \sqrt{\text{card}(E_i)}$$

follows from the inequality

$$\sum_{i=1}^{\infty} x_i \leq \left( \sum_{i=1}^{\infty} \sqrt{x_i} \right)^2 = \sum_{i=1}^{\infty} x_i + \sum_{i<j} \sqrt{x_i x_j}$$

(b) Assume $x \in A$ and $y \notin A$ for $A \in \mathcal{F}(\mu^*)$ and set $E = \{x, y\}$. Then the condition for $A \in \mathcal{F}(\mu^*)$ is

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

for all subsets $E \subseteq X$. If $x \in A$ and $y \notin A$ and $E = \{x, y\}$, then $\mu^*(E) = \sqrt{2}$ and $\mu^*(E \cap A) = \mu^*(E \cap A^c) = 1$, which contradicts (5). Thus $A \in \mathcal{F}(\mu^*)$ can only occur if $A = \phi$ or $A = X$, so that $\mathcal{F}(\mu^*) = \{\phi, X\}$. 