Ma 5051 — Real Variables and Functional Analysis

Solutions for Problem Set #2 due September 17, 2009

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See `m5051hw2.tex` for problem text.

1. Let $\{ A_j : j = 1, 2, \dots \}$ be a sequence of disjoint sets in $\mathcal{M}(\mu^*)$, where $\mathcal{M}(\mu^*)$ means the set of μ^*-measurable sets, and let $B = \bigcup_{j=1}^\infty A_j$. Under these assumptions, the proof of Proposition 1.11 on page 30 contains the inequality

$$\mu^*(E) \;\ge\; \sum_{j=1}^\infty \mu^*(E \cap A_j) \;+\; \mu^*(E \cap B^c)$$

Thus $\mu^*(E \cap B) \ge \sum_{j=1}^\infty \mu^*(E \cap A_j)$ by replacing E by $E \cap B$. Since μ^* is countably subadditive, $\mu^*(E \cap B) = \sum_{j=1}^\infty \mu^*(E \cap A_j)$, which was to be proven.

2. (a) By definition

$$\mu^*(E) = \inf\Big\{ \sum_{i=1}^\infty \mu_0(A_i) \;:\; E \subseteq \bigcup_{i=1}^\infty A_i,\ A_i \in \mathcal{A} \Big\} \tag{1}$$

In general if $E \subseteq \bigcup_{i=1}^\infty A_i$, then $\widetilde{A}_i = A_i - \bigcup_{j=1}^{i-1} A_j \in \mathcal{A}$ where $\{ \widetilde{A}_i \}$ are disjoint, and $\sum_{i=1}^\infty \mu_0(\widetilde{A}_i) \le \sum_{i=1}^\infty \mu_0(A_i)$. Thus it is sufficient in (1) to assume that the A_i are disjoint.

If $\mu^*(E) = \infty$, then $A = \sum_{i=1}^\infty A_i$ for any (disjoint) covering for sets $A_i \in \mathcal{A}$ satisfies $\mu^*(A) = \infty$, which implies $\mu^*(E) \le \mu^*(A) + \epsilon$. If $\mu^*(E) < \infty$, choose a (disjoint) covering $A = \bigcup_{i=1}^\infty A_i$ with $\mu^*(A) \le \mu^*(E) + \epsilon$.

(b) This requires argument in both directions. Note $\mu^*(E) < \infty$. For either direction, choose $Q_n = \bigcup_{j=1}^\infty A_{nj}$ for disjoint $A_{nj} \in \mathcal{A}$ as in part (a) so that $E \subseteq Q_n$ and $\mu^*(Q_n) \le \mu^*(E) + 1/n$. Since the sets $B_n = \bigcap_{i=1}^n Q_i$ are decreasing ($B_{n+1} \subseteq B_n$) and $\mu^*(B_1) = \mu^*(Q_1) < \infty$, it follows from Theorem 1.8 part (d) (page 25) that $\mu^*(B_n) \downarrow \mu^*(B)$ where $B = \bigcap_{i=1}^\infty Q_i = \bigcap_{i=1}^\infty \bigcup_{j=1}^\infty A_{ij}$. Hence $E \subset B$, $B \in \mathcal{A}_{\sigma\delta}$, and $\mu^*(B) = \lim \mu^*(B_n) = \mu^*(E)$.

A set $H \subseteq X$ with $\mu^*(H) = 0$ is called a μ^*-*null set*. The proof of Prop. 1.11 (Carathéodory's Theorem) on page 30 contains a proof that every μ^*-null set is μ^*-measurable. The argument is that, for every subset $E \subset X$,

$$\mu^*(E) \;\le\; \mu^*(E \cap H) + \mu^*(E \cap H^c) \;\le\; \mu^*(E \cap H^c) \;\le\; \mu^*(E)$$

since $\mu^*(E\cap H)\le\mu^*(H)=0$. Hence $\mu^*(E\cap H^c)=\mu^*(E)$ and $\mu^*(E)=\mu^*(E\cap H)+\mu^*(E\cap H^c)$ for every subset $E\subseteq X$, which proves that $H\in\mathcal{M}(\mu^*)$.

For the two directions to be proven: If E is μ^*-measurable, then $H=B-E$ is also μ^*-measurable and, since $E\subseteq B$ and $\mu^*(E)<\infty$, $\mu^*(H)=\mu^*(B)-\mu^*(E)=0$, which was to be proven. Conversely, if $E=B-H$ where $\mu^*(H)=0$, then B and H are both μ^*-measurable. Thus E is measurable since $\mathcal{M}(\mu^*)$ is a σ-algebra, which was to be proven.

(c) If X is σ-finite, then $X=\bigcup_{k=1}^\infty X_k$ where $X_k\in\mathcal{A}$ with $\mu_0(X_k)<\infty$. Since $\mathcal{A}$ is an algebra, we can assume that the X_k are disjoint. Then by part (b) there exist $B_k=\bigcap_{i=1}^\infty\bigcup_{k=1}^\infty A_{kij}$ for $A_{kij}\subseteq X_k$, $A_{kij}\in\mathcal{A}$ such that $\mu^*(E\bigcap X_k)=\mu^*(B_k)$. Since the X_k are disjoint,

$$B\;=\;\bigcup_{k=1}^\infty B_k\;=\;\bigcup_{k=1}^\infty\bigcap_{i=1}^\infty\bigcup_{j=1}^\infty A_{kij}\;=\;\bigcap_{i=1}^\infty\bigcup_{k=1}^\infty\bigcup_{j=1}^\infty A_{kij}$$

implies $B\in\mathcal{A}_{\sigma\delta}$. (*Proof*: If $x\in X$, then $x\in X_{k_0}$ for only one value of k_0, and, since $A_{kij}\subseteq X_k$, the unions over k above reduce to fixing $k=k_0$.)

If E is μ^*-measurable, then $E\bigcap X_k$ is μ^*-measurable with $\mu^*(E\cap X_k)\le\mu(X_k)<\infty$. Thus $E\bigcap X_k=B_k-H_k$ where $B_k\in\mathcal{A}_{\sigma\delta}$ and $\mu^*(H_k)=0$. Thus $E=B-H$ where $B\in\mathcal{A}_{\sigma\delta}$ as above and $H=\bigcup_{k=1}^\infty H_k$ satisfies $\mu^*(H)\le\sum_{k=1}^\infty\mu^*(H_k)=0$, which was to be proven in that direction. If $E=B-H$ where $\mu^*(H)=0$, then E is μ^*-measurable since both B and H are μ^*-measurable, which was to be proven in that direction.

3. This also requires argument in both directions. If E is measurable, then $\mu^*(E)+\mu^*(E^c)=\mu_0(X)$ and $\mu_*(E)=\mu^*(E)$.

Conversely, suppose that $\mu_*(E)=\mu^*(E)$. Then $\mu^*(E)+\mu^*(E^c)=\mu_0(X)<\infty$. By Problem 2, there exist $B_1,B_2\in\mathcal{A}_{\sigma\delta}$ such that

$$\begin{aligned} E\subseteq B_1,&\quad \mu^*(E)=\mu^*(B_1)\\ E^c\subseteq B_2,&\quad \mu^*(E^c)=\mu^*(B_2)\end{aligned}\tag{2}$$

Since $\mu^*(E)+\mu^*(E^c)=\mu_0(X)$, then $\mu^*(B_1)+\mu^*(B_2)=\mu_0(X)$, so that $\mu^*(B_2^c)=\mu_0(X)-\mu^*(B_2)=\mu^*(B_1)$. By (2), $B_2^c\subseteq E\subseteq B_1$. It follows that $\mu^*(B_1-E)\le\mu^*(B_1-B_2^c)=\mu^*(B_1)-\mu^*(B_2^c)=0$. Hence $H=B_1-E$ is a null set and $E=B_1-H$, which implies that E is μ^*-measurable.

4. (a) If $A\in\mathcal{M}$ and $\mu(X)<\infty$, then $\mu(A)+\mu(A^c)=\mu(X)$. By the definition of measurability, $\mu^*(A\cap F)+\mu^*(A^c\cap F)=\mu^*(F)$ for all subsets $F\subset X$, including

$F = E$. Since $\mu^*(E) = \mu(X)$, $\mu^*(A \cap E) \le \mu(A)$, and $\mu^*(A^c \cap E) \le \mu(A^c)$, it follows that $\mu^*(A \cap E) = \mu(A)$ and $\mu^*(A^c \cap E) = \mu(A^c)$, since if (for example) $\mu^*(A \cap E) < \mu(A)$ we could not have $\mu^*(E) = \mu(X)$.

(b) To show that $\mathcal{M}_E$ is a σ-algebra of subsets of E, we need to show (i) $\phi \in \mathcal{M}_E$, (ii) $B \in \mathcal{M}_E$ implies $B^c = E - B \in \mathcal{M}_E$, and (iii) $B_n \in \mathcal{M}_E$ implies $B = \bigcup_{n=1}^\infty B_n \in \mathcal{M}_E$. For (i), note $\phi = \phi \cap E \in \mathcal{M}_E$. For (ii), $B = A \cap E$ $(A \in \mathcal{M})$ implies $E - B = E - A \cap E = A^c \cap E \in \mathcal{M}_E$. For (iii), $B_n = A_n \cap E$ implies $B = A \cap E \in \mathcal{M}_E$ for $A = \bigcup_{n=1}^\infty A_n$, so that $\mathcal{M}_E$ is a σ-algebra.

To show that $\nu = \mu^*$ is a measure on $\mathcal{M}_E$, we need to show (i) $\nu(\phi) = 0$ and (ii) if $\{ A_i \cap E \}$ are disjoint for $A_i \in \mathcal{M}$, then $\mu^*(A \cap E) = \sum_{i=1}^\infty \mu^*(A_i \cap E)$. For (i), $\nu(\phi) = \mu^*(\phi) = 0$. For (ii), even though $\{ A_i \}$ may not be disjoint, we have for $i \ne j$ that $\mu(A_i \cap A_j) = \mu^*(A_i \cap A_j \cap E) = \mu^*\big((A_i \cap E) \cap (A_j \cap E)\big) = \mu^*(\phi) = 0$ by part (a) and $\{ A_i \}$ are disjoint within null sets. Let $N = \sum_{i \ne j} A_i \cap A_j$. Then N is a countable union of null sets so that $\mu(N) = 0$. Also, $(A_i - N) \cap (A_j - N) = A_i \cap A_j - N = \phi$ for $i \ne j$, so that $\{ A_i - N \}$ are disjoint. Thus, if $A = \sum_{i=1}^\infty A_i$,

$$\mu(A) \;=\; \mu(A - N) \;=\; \sum_{i=1}^\infty \mu(A_i - N) \;=\; \sum_{i=1}^\infty \mu(A_i)$$

and by part (a)

$$\nu(A \cap E) \;=\; \mu(A) \;=\; \sum_{i=1}^\infty \mu(A_i) \;=\; \sum_{i=1}^\infty \mu^*(A_i \cap E) \;=\; \sum_{i=1}^\infty \nu(A_i \cap E)$$

Thus $\nu(A)$ is countably additive on $\mathcal{M}_E$, which was to be proven.