Ma 5051 — Real Variables and Functional Analysis

Problem Set #3 — Due September 24, 2009

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1. (Problem 30, page 40) If $E \in \mathcal{L}$ and $0 < m(E) < \infty$, then for any $\alpha < 1$ there exists a nonempty open interval $I = (a, b)$ such that $m(E \cap I) > \alpha m(I)$.

(*Note*: $\mathcal{L} = \mathcal{M}(\lambda^*)$ is the Lebesgue-measurable subsets of R^1 and $m = \lambda^*$; see page 37.)

2. (Like problem 26, page 40) Let $E \in \mathcal{M}(\mu^*)$ for a Borel measure μ on R^1 with $\mu(E) < \infty$ (writing $\mu = \mu^*$ on $\mathcal{M}(\mu^*)$). Prove that, for any $\epsilon > 0$, there exists a finite union of cells $A = \bigcup_{i=1}^n (a_i, b_i]$ such that $\mu(E \bigtriangleup A) < \epsilon$.

3. Let a_i, b_i $(1 \le i \le n)$ be real numbers with $a_i \le b_i$ $(1 \le i \le n)$. Assume that $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i$. Prove that $a_i = b_i$ for $1 \le i \le n$.

4. Let $F(x) = [x]$ be the greatest-integer function on R. Note that $F(x)$ is increasing and right continuous.

(i) Are there any points $a \in R$ for which $\mu_F(\{a\}) > 0$? If so, which points a? What are the corresponding values of $\mu_F(\{a\})$?

(ii) Let $A = \bigcup_{i=1}^5 (\frac{i}{2} - \frac{1}{5}, \frac{i}{2} + \frac{1}{10})$. Find $\mu_F(A)$ and justify your answer. (*Hint*: Be careful!)

5. Let Q be the rationals. Set $(a, b]_Q = \{\, q \in Q : a < q \le b \,\}$ and let $\Gamma_Q = \{\, (a, b]_Q : a, b \in Q \,\}$. Define $\nu(A)$ on Γ_Q by $\nu\big((a, b]_Q\big) = b - a$.

(i) Show that Γ_Q is a semi-ring of subsets of Q and that ν is finitely additive on Γ_Q.

Use ν on Γ_Q to define the outer measure $\nu^*(E)$ for $E \subseteq Q$.

(ii) Show that $\nu^*(\{x\}) = 0$ for all $x \in Q$

(iii) Show that $\nu^*(E) = 0$ for all subsets $E \subseteq Q$.

In particular, it is not true that $\nu^*(A) = \nu(A)$ for all $A \in \Gamma_Q$. While it is true that $\nu^*(A)$ is a measure on the σ-algebra of all subsets of Q, this is not a particularly interesting statement since $\nu^*(A)$ is identically zero.

(iv) At what step or steps do the proofs of Propositions 1.15 or 1.13 break down?