Let \((X, \mathcal{M}, \mu)\) be a measure space. Recall \(\int_A f(x) d\mu = \int I_A(x) f(x) d\mu\) for \(A \in \mathcal{M}\) and \(f \in L^+ \cup L^1\), where \(I_A(x)\) is the indicator function of \(A\).

1. (Problem 20, page 59) Assume \(f_n, g_n, f, g \in L^1\), \(f_n(x) \to f(x)\) and \(g_n(x) \to g(x)\) a.e. as \(n \to \infty\), \(|f_n(x)| \leq g_n(x)\), and \(\int g_n(x) d\mu \to \int g(x) d\mu\). Then show \(\int f_n(x) d\mu \to \int f(x) d\mu\). (Hint: Rework the proof of the dominated convergence theorem.)

2. (Problem 21, page 59) Assume \(f_n, f \in L^1\) and \(f_n(x) \to f(x)\) a.e. Then show that \(\int |f_n - f| d\mu \to 0\) if and only if \(\int |f_n(x)| d\mu \to \int |f(x)| d\mu\). (Hint: Use the previous problem.)

3. (Problem 26, page 60) If \(f \in L^1(R, \mathcal{B}(R), m)\) for Lebesgue measure \(m\) and \(F(x) = \int_{-\infty}^x f(y) dy\), then \(F(x)\) is a continuous function on \(R\).

4. (Problem 28ac, page 60) Compute the following limits and justify the calculations:
   (a) \(\lim_{n \to \infty} \int_0^\infty (1 + (x/n))^{-n} \sin(x/n) dx\)
   (c) \(\lim_{n \to \infty} \int_0^\infty n \sin(x/n) (x(1 + x^2))^{-1} dx\)

5. (Problem 31ac, page 60) Derive the following formulas by expanding part of the integrand into an infinite series and justifying term-by-term integration. Exercise 29 may be useful.
   (a) For \(a > 0\), \(\int_{-\infty}^\infty e^{-x^2} \cos(ax) dx = \sqrt{\pi} e^{-a^2/4}\)
   (c) For \(a > 1\), \(\int_0^\infty x^{a-1} (e^x - 1)^{-1} dx = \Gamma(a) \zeta(a)\), where \(\zeta(a) = \sum_{n=1}^\infty n^{-a}\).