Ma 5051 — Real Variables and Functional Analysis

Problem Set #6 — Due October 15, 2009

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Let $(X, \mathcal{M}, \mu)$ be a measure space. Recall $\int_A f(x)d\mu = \int I_A(x)f(x)d\mu$ for $A \in \mathcal{M}$ and $f \in L^+$, where $I_A(x)$ is the indicator function of A.

1. (Problem 32, page 63) Assume $\mu(X) < \infty$. Let B be the set of all complex-valued measurable functions on X. Define

$$\rho(f,g) = \int_X \frac{|f-g|}{1+|f-g|}\, d\mu$$

for $f, g \in B$. Show that ρ is a metric on B if we identify functions that are equal a.e., and, if $f_n, f \in B$, that $\rho(f_n, f) \to 0$ if and only if $f_n \to f$ in measure.

2. (Problem 34, page 63) Suppose that $|f_n| \le g \in L^1$ and $f_n \to f$ in measure. Show that

(a) $\int f_n d\mu \to \int f d\mu$ as $n \to \infty$

(b) $f_n \to f$ in L^1

3. (Problem 40, page 63) In Egoroff's theorem, the hypothesis " $\mu(X) < \infty$ " can be replaced by " $|f_n| \le g$ for all n where $g \in L^1$ ".

4. (Problem 44, page 64) (Lusin's Theorem) Let $f : [a,b] \to C$ be a complex-valued Lebesgue-measurable function on a closed and bounded interval $[a,b]$. For all $\epsilon > 0$, there exists a compact set $K \subseteq [a,b]$ such that $\mu(K^c) < \epsilon$ and the restriction of $f(x)$ to K is continuous. (*Hint*: Use Egoroff's theorem, Proposition 2.26, and Proposition 1.20.)

5. (Problem 46, page 68) Let $X = Y = [0,1]$. Let $(X, \mathcal{B}(X), \mu)$ be Lebesgue measure and $(Y, \mathcal{B}(Y), \beta)$ be counting measure. Let $D = \{\,(x,x) : x \in [0,1]\,\}$ be the diagonal in $Z = X \times Y$. Show that $A = \iint_Z I_D(z)(\mu \times \beta)(dz)$, $B = \int_Y \left(\int_X I_D(x,y)d\mu\right) d\beta$, and $C = \int_X \left(\int_Y I_D(x,y)d\beta\right) d\mu$ all exist and that A, B, C have three distinct values. (*Hint*: $A = (\mu \times \beta)(D)$ exists since $I_D(x,y)$ is a non-negative Borel function on Z. To compute A, go back to the definition of $\mu \times \beta$ on Borel sets in terms of outer measures.)