Ma 5051 — Real Variables and Functional Analysis

Solutions for Problem Set #6 due October 15, 2009

Prof. Sawyer — Washington University

Let $(X, \mathcal{M}, \mu)$ be a measure space. Recall $\int_A f(x)d\mu = \int I_A(x)f(x)d\mu$ for $A \in \mathcal{M}$ and $f \in L^+$, where $I_A(x)$ is the indicator function of A.

1. (a) Clearly $\rho(f,g) = \rho(g,f)$ and $\rho(f,g) = 0$ if and only if $f = g$ a.e. Thus to prove that ρ is a metric it is sufficient to prove that $\rho(f,h) \le \rho(f,g) + \rho(g,h)$ for all $f,g,h \in B$. Since the function $\phi(A) = A/(1+A)$ is increasing for $A > 0$,

$$\begin{aligned}
\rho(f,h) &= \int_X \frac{|f-h|}{1+|f-h|}\,d\mu \;=\; \int_X \frac{|f-g+g-h|}{1+|f-g+g-h|}\,d\mu \\
&\le \int_X \frac{|f-g|+|g-h|}{1+|f-g|+|g-h|}\,d\mu \\
&= \int_X \frac{|f-g|}{1+|f-g|+|g-h|}\,d\mu \;+\; \int_X \frac{|g-h|}{1+|f-g|+|g-h|}\,d\mu \\
&\le \int_X \frac{|f-g|}{1+|f-g|}\,d\mu \;+\; \int_X \frac{|g-h|}{1+|g-h|}\,d\mu \;=\; \rho(f,g)+\rho(g,h)
\end{aligned}$$

Thus $\rho(f,g)$ satisfies the triangle inequality and is hence a metric.

(b) Assume $\rho(f_n,f) \to 0$. If $|f_n(x) - f(x)| > \epsilon$, then $\epsilon/(1+\epsilon) \le A/(1+A)$ for $A = |f_n(x) - f(x)|$. Hence for all $\epsilon > 0$

$$\begin{aligned}
\mu\big(\{\, x : |f_n(x)-f(x)| > \epsilon \,\}\big) &\le \frac{1+\epsilon}{\epsilon}\int_{\{\,|f_n-f|>\epsilon\,\}} \frac{|f_n-f|}{1+|f_n-f|}d\mu \\
&\le \frac{1+\epsilon}{\epsilon}\int_X \frac{|f_n-f|}{1+|f_n-f|}d\mu \;=\; \frac{1+\epsilon}{\epsilon}\rho(f_n,f) \to 0
\end{aligned}$$

(c) Assume $\mu\big(\{\,|f_n-f|>\epsilon\,\}\big) \to 0$ for all $\epsilon > 0$. Then

$$\begin{aligned}
&\int_X \frac{|f_n-f|}{1+|f_n-f|}\,d\mu \\
&\quad= \int_{X\cap\{\,|f_n-f|>\epsilon\,\}} \frac{|f_n-f|}{1+|f_n-f|}\,d\mu \;+\; \int_{X\cap\{\,|f_n-f|\le\epsilon\,\}} \frac{|f_n-f|}{1+|f_n-f|}\,d\mu \\
&\quad\le \mu\big(\{\,|f_n-f|>\epsilon\,\}\big) \;+\; \frac{\epsilon}{1+\epsilon}\mu(X)
\end{aligned}$$

Thus $\limsup_{n\to\infty} \rho(f_n,f) \le \epsilon\mu(X)$ for all $\epsilon > 0$ and $\rho(f_n,f) \to 0$.

2. Since $\left|\int f_n\,d\mu - \int f\,d\mu\right| = \left|\int (f_n - f)\,d\mu\right| \le \int |f_n - f|d\mu$, part (a) follows from part (b). Thus it is sufficient to prove (b).

Assume $\limsup_{n\to\infty} \int |f_n - f|\,d\mu = A \ge 0$. If $A = 0$, then $\int |f_n - f|\,d\mu \to 0$. If $A > 0$, there exists a sequence $n_k \uparrow \infty$ such that $\lim_{k\to\infty} \int |f_{n_k} - f|\,d\mu = A > 0$.

Since $\{ f_{n_k} \}$ is a subsequence of $\{ f_n \}$, $\mu\{ |f_{n_k} - f| > \epsilon \} \to 0$ for all $\epsilon > 0$. Hence there exists a sequence $k_j \uparrow \infty$ such that $\lim_{j\to\infty} f_{n_{k_j}}(x) = f(x)$ a.e. Since $|f_{n_{k_j}}(x)| \le g(x)$, we conclude $|f(x)| \le g(x)$ and $|f_{n_{k_j}}(x) - f(x)| \le 2g(x)$. Then, by dominated convergence, $\lim_{j\to\infty} \int |f_{n_{k_j}} - f|\,d\mu = 0$. Since $\{ f_{n_{k_j}} \}$ is a subsequence of $\{ f_{n_k} \}$ and $\lim_{k\to\infty} \int |f_{n_k} - f|\,d\mu = A$, this implies $A = 0$. Hence $\int |f_n - f|d\mu \to 0$.

3. Since $f_n(x) \to f(x)$ a.e., then $|f(x)| \le g(x)$ and $\sup_{n\ge1} |f_n(x) - f(x)| \le 2g(x)$. As in the proof of Egoroff's theorem, let $A_n(\epsilon) = \{ \sup_{m\ge n} |f_m - f| > \epsilon \}$. Then $A_n(\epsilon) \downarrow \phi$ for fixed $\epsilon > 0$ and

$$\mu\big(A_1(\epsilon)\big) \;\le\; \mu\{ 2g > \epsilon \} \;\le\; \frac{2}{\epsilon}\int g(x)\,d\mu < \infty$$

If $\mu\big(A_1(\epsilon)\big) < \infty$, $A_n(\epsilon) \downarrow \phi$ implies $\mu\big(A_n(\epsilon)\big) \downarrow 0$ for each $\epsilon > 0$. Thus, as in the proof of Egoroff's theorem in the text, can choose $n_k \uparrow \infty$ such that $\mu\big(A_{n_k}(1/k)\big) < \epsilon/2^k$. Then $A = \bigcup_{k=1}^{\infty} A_{n_k}(1/k)$ implies $\mu(A) < \epsilon$ and $f_n(x) \to f(x)$ uniformly on A^c.

4. By Proposition 2.26, there exist continuous functions $\phi_n(x)$ on $[a, b]$ such that $\int |f(x) - \phi_n(x)|\,d\mu < 1/2^n$. Then

$$\int_a^b \sum_{n=1}^{\infty} |f(x) - \phi_n(x)|\,dx \;=\; \sum_{n=1}^{\infty} \int_a^b |f(x) - \phi_n(x)|\,dx \;<\; 1$$

Since integrable functions are finite a.e., this implies $\phi_n(x) \to f(x)$ a.e. Hence by Egoroff's theorem, there exists a measurable set E with $m(E^c) < \epsilon/2$ such that $\phi_n(x) \to f(x)$ uniformly on E. By Proposition 1.20, we can choose a compact set $K \subseteq E$ such that $m(E - K) < \epsilon/2$. Since $K^c = (E - K) \cup E^c$, we conclude $m(K^c) < \epsilon$ and $\phi_n(x) \to f(x)$ uniformly on K.

5. Here μ is Lebesgue measure and β is counting measure on $[0, 1]$. Since, for fixed y, $I_D(x, y) = 0$ except for $x = y$, $\int_X I_D(x, y)d\mu(x) = \mu(\{ y \}) = 0$ for all y, and $B = \int_Y \big(\int_X I_D(x, y)d\mu(x)\big)\,d\beta(y) = 0$. Similarly, for fixed x, $\int_Y I_D(x, y)d\beta(y) = \beta(\{ x \}) = 1$ and $C = \int_X \big(\int_Y I_D(x, y)d\beta\big)\,d\mu = 1$.

Finally, $\int I_D(z)d(\mu\times\beta)(z) = (\mu\times\beta)(D)$ where, by definition,

$$(\mu\times\beta)(D) = \inf\left\{ \sum_{k=1}^{\infty} \mu(A_k)\beta(B_k) :\ D \subseteq \bigcup_{k=1}^{\infty}(A_k\times B_k),\ A_k \in \mathcal{B}(R),\ B_k \subseteq [0, 1] \right\}$$

(Remark: The σ-algebra $\mathcal{M}_2$ on which counting measure β is defined is ambiguous. As defined on page 25 in the text, $\mathcal{M}_2$ is the set of all countable or co-countable (complements of countable) sets in $[0, 1]$. If it helps your proof, you can assume $B_k \in \mathcal{M}_2$ above.)

Let $D \subseteq \bigcup_{k=1}^{\infty}(A_k \times B_k)$ be one of the coverings above. If we can show that this implies that $\mu(A_k) > 0$ and $\beta(B_k) = \infty$ for at least one value of k, then $(\mu \times \beta)(D) = \infty$ and $A = \iint_Z I_D(z)(\mu \times \beta)(dz) = \infty$.

For each $x \in [0, 1]$, $(x, x) \in A_k \times B_k$ for at least one value of k. Let K_1 be the union of the A_k with $\mu(A_k) = 0$. Since the union is countable, $\mu(K_1) = 0$. Let K_2 be the union of the B_k with $\beta(B_k) < \infty$; i.e, such that B_k is finite. Then K_2 is countable and $\mu(K_2) = 0$. Since $\mu([0, 1] - (K_1 \cup K_2)) = 1 - \mu(K_1 \cup K_2) = 1$, there exists at least one value $x \notin K_1 \cup K_2$.

For this x, assume $(x, x) \in A_k \times B_k$, so that $x \in A_k$ and $x \in B_k$. Since $x \notin K_1$ (that is, x is not in the union of the A_k with $\mu(A_k) = 0$), $\mu(A_k) > 0$. Since $x \notin K_2$ (that is, x is not in the union of the B_k with $\beta(B_k) < \infty$), $\beta(B_k) = \infty$. Thus $\mu(A_k)\beta(B_k) = \infty$, which completes the proof of $(\mu \times \beta)(D) = \infty$ and $A = \iint_Z I_D(z)(\mu \times \beta)(dz) = \infty$. In particular, the values A, B, C are $\infty, 0, 1$.