Let \((X, \mathcal{M}, \mu)\) be a measure space. Recall \(\int_A f(x)\,d\mu = \int I_A(x)f(x)\,d\mu\) for \(A \in \mathcal{M}\) and \(f \in L^+\), where \(I_A(x)\) is the indicator function of \(A\).

1. (a) Clearly \(\rho(f, g) = \rho(g, f)\) and \(\rho(f, g) = 0\) if and only if \(f = g\) a.e. Thus to prove that \(\rho\) is a metric it is sufficient to prove that \(\rho(f, h) \le \rho(f, g) + \rho(g, h)\) for all \(f, g, h \in B\). Since the function \(\phi(A) = A/(1 + A)\) is increasing for \(A > 0\),

\[
\rho(f, h) = \int_X \frac{|f - h|}{1 + |f - h|} \,d\mu = \int_X \frac{|f - g + g - h|}{1 + |f - g + g - h|} \,d\mu
\]

\[
\le \int_X \frac{|f - g|}{1 + |f - g| + |g - h|} \,d\mu + \int_X \frac{|g - h|}{1 + |f - g| + |g - h|} \,d\mu
\]

\[
= \int_X \frac{|f - g|}{1 + |f - g| + |g - h|} \,d\mu + \int_X \frac{|g - h|}{1 + |f - g| + |g - h|} \,d\mu = \rho(f, g) + \rho(g, h)
\]

Thus \(\rho(f, g)\) satisfies the triangle inequality and is hence a metric.

(b) Assume \(\rho(f_n, f) \to 0\). If \(|f_n(x) - f(x)| > \epsilon\), then \(\epsilon/(1 + \epsilon) \le A/(1 + A)\) for \(A = |f_n(x) - f(x)|\). Hence for all \(\epsilon > 0\)

\[
\mu\left(\{ x : |f_n(x) - f(x)| > \epsilon \} \right) \le \frac{1 + \epsilon}{\epsilon} \int_{\{ |f_n - f| > \epsilon \}} \frac{|f_n - f|}{1 + |f_n - f|} \,d\mu
\]

\[
\le \frac{1 + \epsilon}{\epsilon} \int_X \frac{|f_n - f|}{1 + |f_n - f|} \,d\mu = \frac{1 + \epsilon}{\epsilon} \rho(f_n, f) \to 0
\]

(c) Assume \(\mu(\{|f_n - f| > \epsilon\}) \to 0\) for all \(\epsilon > 0\). Then

\[
\int_X \frac{|f_n - f|}{1 + |f_n - f|} \,d\mu
\]

\[
= \int_{X \cap \{ |f_n - f| > \epsilon \}} \frac{|f_n - f|}{1 + |f_n - f|} \,d\mu + \int_{X \cap \{ |f_n - f| \le \epsilon \}} \frac{|f_n - f|}{1 + |f_n - f|} \,d\mu
\]

\[
\le \mu(\{ |f_n - f| > \epsilon \}) + \frac{\epsilon}{1 + \epsilon} \mu(X)
\]

Thus \(\limsup_{n \to \infty} \rho(f_n, f) \le \epsilon \mu(X)\) for all \(\epsilon > 0\) and \(\rho(f_n, f) \to 0\).
2. Since $|\int f_n \, d\mu - \int f \, d\mu| = |\int (f_n - f) \, d\mu| \leq \int |f_n - f| \, d\mu$, part (a) follows from part (b). Thus it is sufficient to prove (b).

Assume $\lim sup_{n \to \infty} \int |f_n - f| \, d\mu = A \geq 0$. If $A = 0$, then $\int |f_n - f| \, d\mu \to 0$. If $A > 0$, there exists a sequence $n_k \uparrow \infty$ such that $\lim_{k \to \infty} \int |f_{n_k} - f| \, d\mu = A > 0$.

Since $\{f_{n_k}\}$ is a subsequence of $\{f_n\}$, $\mu\{|f_{n_k} - f| > \epsilon\} \to 0$ for all $\epsilon > 0$. Hence there exists a sequence $k_j \uparrow \infty$ such that $\lim_{j \to \infty} f_{n_{k_j}}(x) = f(x)$ a.e. Since $|f_{n_{k_j}}(x)| \leq g(x)$, we conclude $|f(x)| \leq g(x)$ and $|f_{n_{k_j}}(x) - f(x)| \leq 2g(x)$. Then, by dominated convergence, $\lim_{j \to \infty} \int |f_{n_{k_j}} - f| \, d\mu = 0$. Since $\{f_{n_{k_j}}\}$ is a subsequence of $\{f_{n_k}\}$ and $\lim_{k \to \infty} \int |f_{n_k} - f| \, d\mu = A$, this implies $A = 0$. Hence $\int |f_n - f| \, d\mu \to 0$.

3. Since $f_n(x) \to f(x)$ a.e., then $|f(x)| \leq g(x)$ and $\sup_{n \geq 1} |f_n(x) - f(x)| \leq 2g(x)$. As in the proof of Egoroff’s theorem, let $A_n(\epsilon) = \{\sup_{m \geq n} |f_m - f| > \epsilon\}$. Then $A_n(\epsilon) \downarrow \phi$ for fixed $\epsilon > 0$ and

$$\mu(A_1(\epsilon)) \leq \mu(2g > \epsilon) \leq \frac{2}{\epsilon} \int g(x) \, d\mu < \infty$$

If $\mu(A_1(\epsilon)) < \infty$, $A_n(\epsilon) \downarrow \phi$ implies $\mu(A_n(\epsilon)) \downarrow 0$ for each $\epsilon > 0$. Thus, as in the proof of Egoroff’s theorem in the text, can choose $n_k \uparrow \infty$ such that $\mu(A_{n_k}(1/k)) < \epsilon/2^k$. Then $A = \bigcup_{k=1}^{\infty} A_{n_k}(1/k)$ implies $\mu(A) < \epsilon$ and $f_n(x) \to f(x)$ uniformly on $A^c$.

4. By Proposition 2.26, there exist continuous functions $\phi_n(x)$ on $[a, b]$ such that $\int |f(x) - \phi_n(x)| \, dx < 1/2^n$. Then

$$\int_{a}^{b} \sum_{n=1}^{\infty} |f(x) - \phi_n(x)| \, dx = \sum_{n=1}^{\infty} \int_{a}^{b} |f(x) - \phi_n(x)| \, dx < 1$$

Since integrable functions are finite a.e., this implies $\phi_n(x) \to f(x)$ a.e. Hence by Egoroff’s theorem, there exists a measurable set $E$ with $m(E^c) < \epsilon/2$ such that $\phi_n(x) \to f(x)$ uniformly on $E$. By Proposition 1.20, we can choose a compact set $K \subseteq E$ such that $m(E - K) < \epsilon/2$. Since $K^c = (E - K) \cup E^c$, we conclude $m(K^c) < \epsilon$ and $\phi_n(x) \to f(x)$ uniformly on $K$.

5. Here $\mu$ is Lebesgue measure and $\beta$ is counting measure on $[0, 1]$. Since, for fixed $y$, $I_D(x, y) = 0$ except for $x = y$, $\int X I_D(x, y) \, d\mu(x) = \mu(\{y\}) = 0$ for all $y$, and $B = \int Y \left(\int X I_D(x, y) \, d\mu(x)\right) \, d\beta(y) = 0$. Similarly, for fixed $x$, $\int Y I_D(x, y) \, d\beta(y) = \beta(\{x\}) = 1$ and $C = \int X \left(\int Y I_D(x, y) \, d\beta\right) \, d\mu = 1$.

Finally, $\int I_D(z) \, d(\mu \times \beta)(z) = (\mu \times \beta)(D)$ where, by definition,

$$(\mu \times \beta)(D) = \inf \left\{ \sum_{k=1}^{\infty} \mu(A_k) \beta(B_k) : D \subseteq \bigcup_{k=1}^{\infty} (A_k \times B_k), \ A_k \in \mathcal{B}(R), \ B_k \subseteq [0, 1] \right\}$$
(Remark: The $\sigma$-algebra $\mathcal{M}_2$ on which counting measure $\beta$ is defined is ambiguous. As defined on page 25 in the text, $\mathcal{M}_2$ is the set of all countable or co-countable (complements of countable) sets in $[0,1]$. If it helps your proof, you can assume $B_k \in \mathcal{M}_2$ above.)

Let $D \subseteq \bigcup_{k=1}^{\infty} (A_k \times B_k)$ be one of the coverings above. If we can show that this implies that $\mu(A_k) > 0$ and $\beta(B_k) = \infty$ for at least one value of $k$, then $(\mu \times \beta)(D) = \infty$ and $A = \int \int_Z I_D(z)(\mu \times \beta)(dz) = \infty$.

For each $x \in [0,1]$, $(x,x) \in A_k \times B_k$ for at least one value of $k$. Let $K_1$ be the union of the $A_k$ with $\mu(A_k) = 0$. Since the union is countable, $\mu(K_1) = 0$. Let $K_2$ be the union of the $B_k$ with $\beta(B_k) < \infty$; i.e., such that $B_k$ is finite. Then $K_2$ is countable and $\mu(K_2) = 0$. Since $\mu([0,1] - (K_1 \cup K_2)) = 1 - \mu(K_1 \cup K_2) = 1$, there exists at least one value $x \notin K_1 \cup K_2$.

For this $x$, assume $(x,x) \in A_k \times B_k$, so that $x \in A_k$ and $x \in B_k$. Since $x \notin K_1$ (that is, $x$ is not in the union of the $A_k$ with $\mu(A_k) = 0$), $\mu(A_k) > 0$. Since $x \notin K_2$ (that is, $x$ is not in the union of the $B_k$ with $\beta(B_k) < \infty$), $\beta(B_k) = \infty$. Thus $\mu(A_k) \beta(B_k) = \infty$, which completes the proof of $(\mu \times \beta)(D) = \infty$ and $A = \int \int_Z I_D(z)(\mu \times \beta)(dz) = \infty$. In particular, the values $A, B, C$ are $\infty, 0, 1$. 