Ma 5051 — Real Variables and Functional Analysis

Solutions for Problem Set #8 due November 12, 2009

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(With Matt Wallace)

1. Let $E \in \mathcal{M}_1 \otimes \mathcal{M}_2$, where $\mathcal{M}_1 \otimes \mathcal{M}_2$ is the product σ-algebra. By assumption, $\nu_1(A) = \int_A g_1(x)\mu_1(dx)$ for $A \in \mathcal{M}_1$ and $\nu_2(B) = \int_B g_2(y)\mu_2(dy)$ for $B \in \mathcal{M}_1$ where $g_1(x) = (d\nu_1/d\mu_1)(x) \geq 0$ a.e. (μ_1) and $g_2(y) = (d\nu_1/d\mu_2)(y) \geq 0$ a.e. (μ_2) since $\nu_1, \nu_2, \mu_1, \mu_2$ are positive measures. Thus by Tonelli's Theorem (page 67)

$$\begin{aligned}(\nu_1 \times \nu_2)(E) &= \int_{X_1 \times X_2} I_E(x_1, x_2)\, d(\nu_1 \times \nu_2)(x_1, x_2) &(1)\\ &= \int_{X_1} \left(\int_{X_2} I_E(x_1, x_2) d\nu_2(x_2) \right) d\nu_1(x_1)\\ &= \int_{X_1} \left(\int_{X_2} I_E(x_1, x_2) g_2(x_2) d\mu_2(x_2) \right) g_1(x_1) d\mu_1(x_1)\\ &= \int_{X_1} \left(\int_{X_2} I_E(x_1, x_2) g_1(x_1) g_2(x_2) d\mu_2(x_2) \right) d\mu_1(x_1)\end{aligned}$$

and by Tonelli's Theorem a second time

$$\begin{aligned}(\nu_1 \times \nu_2)(E) &= \int_{X_1 \times X_2} I_E(x_1, x_2) g_1(x_1) g_2(x_2)\, d(\mu_1 \times \mu_2)(x_1, x_2)\\ &= \int_E g_1(x_1) g_2(x_2)\, d(\mu_1 \times \mu_2)(x_1, x_2) &(2)\end{aligned}$$

This implies that $(\nu_1 \times \nu_2) \ll (\mu_1 \times \mu_2)$ and the Radon-Nikodym derivative is $d(\nu_1 \times \nu_2)/d(\mu_1 \times \mu_2) = g_1(x_1)g_2(x_2) = (d\nu_1/d\mu_1)(x_1)(d\nu_2/d\mu_2)(x_2)$.

Remark. Since all these measures are σ-finite, it is sufficient to verify (2) on a generating semi-ring of sets E. Since the set of measurable rectangles $\{\, A \times B : A \in \mathcal{M}_1, B \in \mathcal{M}_2 \,\}$ is a generating semi-ring for $\mathcal{M}_1 \otimes \mathcal{M}_2$, it is sufficient to assume that $E = A \times B$ in (1) is a measurable rectangle. This simplifies the proof slightly.

2. (a) If $\lambda(E) = \nu(E) + \mu(E) = 0$, then $\nu(E) = \mu(E) = 0$ and $\nu \ll \lambda$. If $\mu(E) = 0$, then $\nu(E) = 0$ since $\nu \ll \mu$ and hence $\lambda(E) = \nu(E) + \mu(E) = 0$, which implies $\lambda \ll \mu$.

(b) Assume $\nu(E) = \int_E g\, d\mu$. Since ν and μ are positive and σ-finite, $0 \leq g(x) < \infty$ a.e. μ. Then $\lambda(E) = \int_E (1+g)\, d\mu$. Hence $\int h\, d\lambda = \int h(1+g)\, d\mu$ for all measurable $h(x) \geq 0$. Setting $h = (1+g)^{-1} I_E$, $\mu(E) = \int_E (1+g)^{-1}\, d\lambda$ and

$$\nu(E) = \int_E g\, d\mu = \int_E \frac{g}{1+g}\, d\lambda = \int_E f\, d\lambda$$

for $f = d\nu/d\lambda = g/(1+g)$. Since $0 \le g < \infty$, $0 \le f(x) < 1$ a.e. μ.

(c) $f = g/(1+g)$ implies $f + fg = g$ and $g = f/(1-f)$, so $d\nu/d\mu = g = f/(1-f)$.

3. Here μ, ν are finite positive measures on $(X, \mathcal{M})$ with $\nu(E) = \int f\,d\mu$ for some $\mathcal{M}$-measurable functions $f(x) \ge 0$. Restrict both measures μ, ν to $(X, \mathcal{A})$ for a σ-algebra $\mathcal{A} \subseteq \mathcal{M}$. It may not be true that $\nu(E) = \int_E f\,d\mu$ for $f \in L^1(X, \mathcal{A}, \mu)$, but $\mu(E) = 0$ for $E \in \mathcal{A}$ still implies $\nu(E) = 0$ since $\mathcal{A} \subseteq \mathcal{M}$. It then follows from the Radon-Nikodym theorem that $\mu(E) = \int_E g\,d\mu$ for $E \in \mathcal{A}$ and $g \in L^1(X, \mathcal{A}, \mu)$, where now we are guaranteed that $g(x)$ is $\mathcal{A}$-measurable.

If $\mu(E) = \int_E g_1\,d\mu = \int_E g_2\,d\mu$ for all $E \in \mathcal{A}$ and $g_1, g_2 \in L^1(X, \mathcal{A}, \mu)$, then $\int_E (g_1 - g_2) d\mu = 0$ for all $E \in \mathcal{A}$ and by standard arguments for $(X, \mathcal{A}, \mu)$, $g_1(x) = g_2(x)$ a.e. μ.

4. Assume $d\nu = f d\mu$ for some positive measure μ where $f \in L^1(X, \mathcal{M}, \mu)$. Then (by definition) $d|\nu| = |f| d\mu$ and $\nu(X) = \int f(x)\,d\mu = |\nu|(X) = \int |f(x)|\,d\mu = 0$. Thus $\int \big(|f(x)| - f(x)\big) d\mu = 0$ and

$$\operatorname{Re} \int \big(|f(x)| - f(x)\big)\,d\mu \;=\; \int \big(|f(x)| - \operatorname{Re} f(x)\big)\,d\mu \;=\; 0$$

In general, $\int g(x)\,d\mu(x) = 0$ for $g(x) \ge 0$ and a positive measure μ implies $g(x) = 0$ a.e. μ. Since $g(x) = |f(x)| - \operatorname{Re} f(x) \ge 0$ for all x, it follows that $\operatorname{Re} f(x) = |f(x)|$ a.e. μ. Since $|f(x)| = |\operatorname{Re} f(x) + i \operatorname{Im} f(x)|$, this also implies $\operatorname{Im} f(x) = 0$ a.e. μ. Thus $f(x) = |f(x)|$ a.e. μ. and $\nu = |\nu|$.

5. Assume $\nu(E) = \int_E g(x)\,d\mu$ for complex $g \in L^1(X, \mathcal{M}, \mu)$. Then (by definition) $|\nu|(E) = \int_E |g(x)|\,d\mu$ and it is sufficient to prove

$$\int_E |g|\,d\mu \;=\; \sup \left\{ \left| \int_E f(x) g(x)\,d\mu \right| \;:\; |f(x)| \le 1 \text{ on } E \right\} \tag{3}$$

Since $|\int_E fg\,d\mu \le \int_E |fg|\,d\mu \le \int_E |g|\,d\mu$ for complex functions f, g with $|f(x)| \le 1$, the right-hand side of (3) is less than or equal to the left-hand side. Define $f(x) = |g(x)|/g(x)$ if $g(x) \ne 0$ and $f(x) = 0$ if $g(x) = 0$. Then $\int_E f(x) g(x)\,d\mu = \int_E |g(x)|\,d\mu$ and the two sides of (3) are identical.