Ma 5051 — Real Variables and Functional Analysis
Problem Set #9 — Due November 19, 2009
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The measure $m(E)$ below is Lebesgue measure on $B(R^n)$.

1. (Like Problem 22, page 100) Let $Hf(x)$ be the Hardy-Littlewood maximal function for $f \in L^1(R^n)$ with $\int_{R^n} |f(y)| dm > 0$. Prove that there exists $C > 0$ and $R < \infty$ such that $Hf(x) \geq C/|x|^n$ for $|x| > R$.
   
   Use this to show that $m[\{x : Hf(x) > \alpha\}] \geq C_1/\alpha$ for some $C_1 > 0$ and $\alpha < \alpha_0$ for some $\alpha_0 > 0$, and hence the rate of decay in $\alpha$ in the Hardy-Littlewood maximal theorem (Theorem 3.17) is sharp.
   
   (Hint: You can use the results for polar coordinates in Section 2.7.)

2. (Problem 24, page 100) Assume $f \in L^1_{loc}(R^n)$ and $f$ is continuous at $x$. Show that $x$ is in the Lebesgue set of $f$.

3. (Like Problem 25, page 100) Let $E$ be a Borel subset of $R^n$. The density of $E$ at a point $x$ is defined by

   $$D_E(x) = \lim_{r \to 0} \frac{m(E \cap B(r,x))}{m(B(r,x))}$$

   whenever the limit exists.

   (a) Show that $D_E(x) = 1$ a.e. for $x \in E$ and $D_E(x) = 0$ a.e. for $x \in E^c$.

   (b) Find an example of $E \subseteq R^2$ and $x \in R^2$ such that $D_E(x) = 1/4$.

4. (Like Problem 32, page 108) Assume $F_n \in BV(R)$. Assume $F_n(x) \to F(x)$ for all $x \in R$ for some function $F(y)$. Prove that for all $y \in R$

   $$V_F(y) \leq \liminf_{n \to \infty} V_{F_n}(y)$$

   ($V_F(y)$ is called $T_F(y)$ in the text.)

5. (Problem 33, page 108) Suppose $F(y) : R \to R$ is increasing. Prove that $F(b) - F(a) \geq \int_a^b F'(x)dx$. 