Ma 5051 — Real Variables and Functional Analysis

Problem Set #9 — Due November 19, 2009

Prof. Sawyer — Washington University

The measure $m(E)$ below is Lebesgue measure on $\mathcal{B}(R^n)$.

1. (Like Problem 22, page 100) Let $Hf(x)$ be the Hardy-Littlewood maximal function for $f \in L^1(R^n)$ with $\int_{R^n} |f(y)|dm > 0$. Prove that there exists $C > 0$ and $R < \infty$ such that $Hf(x) \geq C/|x|^n$ for $|x| > R$.

Use this to show that $m\big[\{\, x : Hf(x) > \alpha \,\}\big] \geq C_1/\alpha$ for some $C_1 > 0$ and $\alpha < \alpha_0$ for some $\alpha_0 > 0$, and hence the rate of decay in α in the Hardy-Littlewood maximal theorem (Theorem 3.17) is sharp.

(*Hint*: You can use the results for polar coordinates in Section 2.7.)

2. (Problem 24, page 100) Assume $f \in L^1_{loc}(R^n)$ and f is continuous at x. Show that x is in the Lebesgue set of f.

3. (Like Problem 25, page 100) Let E be a Borel subset of R^n. The *density* of E at a point x is defined by

$$D_E(x) \;=\; \lim_{r \to 0} \frac{m(E \cap B(r,x))}{m(B(r,x))}$$

whenever the limit exists.

(a) Show that $D_E(x) = 1$ a.e. for $x \in E$ and $D_E(x) = 0$ a.e. for $x \in E^c$.

(b) Find an example of $E \subseteq R^2$ and $x \in R^2$ such that $D_E(x) = 1/4$.

4. (Like Problem 32, page 108) Assume $F_n \in BV(R)$. Assume $F_n(x) \to F(x)$ for all $x \in R$ for some function $F(y)$. Prove that for all $y \in R$

$$V_F(y) \;\leq\; \liminf_{n \to \infty} V_{F_n}(y)$$

($V_F(y)$ is called $T_F(y)$ in the text.)

5. (Problem 33, page 108) Suppose $F(y) : R \to R$ is increasing. Prove that $F(b) - F(a) \geq \int_a^b F'(x)dx$.