Ma 5051 — Real Variables and Functional Analysis

Solutions for Problem Set #9 due November 19, 2009

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The measure $m(E)$ below is Lebesgue measure on $\mathcal{B}(R^n)$.

1. (a) Choose a constant K such that $\int_{|x|\le K}|f(y)|dm \ge (1/2)\int_{R^n}|f(y)|dm$. Recall that $m(B(r,x)) = C_n r^n$ for $B(r,x)\subseteq R^n$. If $|x|\ge K$ and $K_x = |x|+K$, the ball $B(K_x,x)\supseteq B(K,0)$ and

$$\begin{aligned} Hf(x) \;&\ge\; \frac{1}{m(B(K_x,x))}\int_{B(K_x,x)}|f(y)|dm \;\ge\; \frac{1}{C_n\,(K_x)^n}\int_{B(K,0)}|f(y)|dm \\ &\ge\; \frac{C_1}{|x|^n}\left(\frac{|x|}{|x|+K}\right)^n (1/2)\int_{R^n}|f(y)|dm \;\ge\; C/|x|^n \end{aligned}$$

for some $C>0$, since $|x|/(|x|+K)\ge 1/2$ if $|x|\ge K$.

(b) By part (a), $Hf(x) > \alpha$ whenever $|x|\ge K$ and $C/|x|^n > \alpha$, or when $K\le |x|$ and $|x| < (C/\alpha)^{1/n}$. Thus

$$\begin{aligned} m\big[\{\,x : Hf(x) > \alpha\,\}\big] \;&\ge\; m\big[\{\,x : K \le |x| \le (C/\alpha)^{1/n}\,\}\big] \\ &=\; C_n\big((C/\alpha) - K^n\big) \;=\; C_n(C/\alpha)\left(1 - \frac{K^n}{C/\alpha}\right) \;\ge\; C_2/\alpha \end{aligned}$$

for $C_2 = C_n C/2$ provided that α is sufficiently small so that $C/\alpha > (2K)^n$, or so that $\alpha < \alpha_0 = C/(2K)^n$.

2. By definition, $x \in L_f$ if

$$\lim_{r\to 0}\frac{1}{m(B(r,x))}\int_{B(r,x)}|f(y)-f(x)|dy \;=\; 0$$

If $|f(y)-f(x)| < \epsilon$ whenever $|y-x| < r_0$, then the integral above is bounded by $\epsilon\, m(B(r,x))$ for $r < r_0$ and the expression inside the limit is less than ϵ. Thus $\lim_{y\to x} f(y) = f(x)$ implies $x\in L_f$.

3. By the Lebesgue differentiation theorem in the form Theorem 3.18,

$$D_E(x) \;=\; \lim_{r\to 0}\frac{1}{m(B(r,x))}\int_{B(r,x)} I_E(y)dy \;=\; \lim_{r\to 0}\frac{m(E\cap B(r,x))}{m(B(r,x))} \;=\; I_E(x) \text{ a.e.}$$

In particular, $D_E(x) = 1$ a.e. for $x\in E$ and $D_E(x) = 0$ a.e. for $x \in E^c$.

Let $E = [0,1]\times[0,1] = \{\,(x,y) : 0 \le x,y \le 1\,\}$ and set $x_0 = 0$. Then $m(E\cap B(r,0)) = (1/4)m(B(r,0))$ whenever $r<1$ and $D_E(x_0) = 1/4$.

4. Define

$$V_F(y,\pi(y)) = \sum_{j=1}^{k} |F(x_j) - F(x_{j-1})|$$

for partitions $\pi(y) = \{x_j : 0 \le j \le k,\ -\infty < x_0 < x_1 < \ldots < x_k = y\}$. Then $V_F(y) = T_F(y) = \sup_{\pi(y)} V_F(y,\pi(y))$. By assumption, $F_n(x) \to F(x)$ for all x, so that, for each fixed partition $\pi(y)$, $V_{F_n}(y,\pi(y)) \to V_F(y,\pi(y))$. Since $V_{F_n}(y) = \sup_{\pi(y)} V_{F_n}(y,\pi(y))$ for each n,

$$V_F(y,\pi(y)) \ = \ \lim_{n\to\infty} V_{F_n}(y,\pi(y)) \ \le \ \liminf_{n\to\infty} V_{F_n}(y)$$

Since this holds for all partitions $\pi(y)$, $V_F(y) \le \liminf_{n\to\infty} V_{F_n}(y)$.

5. Let $G(y) = F(y+)$ is increasing and right-continuous (see Theorem 3.23 on page 101). Thus there exists a Borel measure μ_G such that $\mu_G((a,b]) = G(b) - G(a)$ for all $a, b \in R$, $a < b$.

By the Lebesgue-Radon-Nikodym Theorem (page 90), $d\mu_G = d\lambda + g(x)dx$ where $\lambda \ge 0$, $d\lambda \perp dx$, $g(x) \ge 0$ a.e., and dx is one-dimensional Lebesgue measure. In particular

$$G(b) - G(a) \ = \ \mu_G((a,b]) \ = \ \lambda((a,b]) + \int_a^b g(x)dx \ \ge \ \int_a^b g(x)dx$$

By the general Lebesgue differentiation theorem (Thm 3.33, page 98), $G'(x) = g(x)$ a.e. (and $\lambda'(x) = 0$ a.e.), so that

$$G(b) - G(a) \ \ge \ \int_a^b G'(x)dx$$

In general, if $G(y)$ is any increasing function such that $G(y) \ge h(y)$ where $h(y)$ is a continuous function, then, by considering points of continuity y of $G(y)$ and then taking increasing and decreasing limits, $G(y\pm) \ge h(y)$ for all y. By Theorem 3.23 (page 101), $F'(x) = G'(x)$ a.e. Putting this together

$$G(b\pm) - G(a\pm) \ \ge \ G(b-) - G(a+) \ \ge \ \int_a^b F'(x)dx$$

Finally, since $G(b-) \le F(b) \le G(b)$ and $G(a-) \le F(a) \le G(a)$, we conclude for all $a < b$ that

$$F(b) - F(a) \ \ge \ G(b-) - G(a) \ \ge \ \int_a^b F'(x)dx$$

Remark: You can also use step functions based on difference quotients, Fatou's lemma, and Theorem 3.23 to conclude $F(b) - F(a) \ge \int_a^b F'(x)dx$. Indeed, from this point of view, the use of the Radom-Nikodym theorem plus the Lebesgue differention theorem is overkill.