Ma 5051 — Real Variables and Functional Analysis

Take-Home Midterm — Due October 22, 2009

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Let $(X, \mathcal{M}, \mu)$ be a measure space. Recall $\int_A f(x)d\mu = \int I_A(x)f(x)d\mu$ for $A \in \mathcal{M}$ and $f \in L^+$, where $I_A(x)$ is the indicator function of A.

1. (a) Evaluate the limit and justify the calculation:

$$\lim_{n\to\infty} \int_0^n x^k(1 - n^{-1}x)^n \, dx$$

(b) Evaluate the limit and justify the calculation:

$$\lim_{n\to\infty} \sqrt{n} \int_0^1 (1 + nx^2)(1 + x^2)^{-n} \, dx$$

2. Expand the following as a power series in a and justify any interchange of integrals and sums:

$$\int_0^\infty e^{-y^2} \sin(ay) \, dy$$

3. (Like Problem 33 page 63) If $f_n(x) \geq 0$ are measurable and $f_n \to f$ in measure for a measure μ, show that $\int f(x) \, d\mu \leq \liminf_{n\to\infty} \int f_n(x) \, d\mu$.

4. For $X = [0, 1]$, let $\mu = m + 2\delta_a$ for some $a \in (0, 1)$ where m is Lebesgue measure and δ_a is the delta measure based at a. (That is, $\delta_a(E) = 1$ if $a \in E$ and $\delta_a(E) = 0$ if $a \notin E$.) Thus $\mu(X) = 3$. Evaluate the following limit and justify your calculations:

$$\lim_{n\to\infty} \int_0^1 \int_0^1 e^{-n(x-y)^2} d\mu(x)d\mu(y)$$

(*Warning*: The measure $\mu(A)$ may have atoms.)

5. (Like Problem 50 page 69) Assume that $(X, \mathcal{M}, \mu)$ is a σ-finite measure and assume $f \in L^+(X, \mathcal{M})$. Let m be Lebesgue measure on $R^+ = [0, \infty)$ and set

$$G_f \;=\; \{\, (x, y) \in (X \times [0, \infty)) : 0 \leq y < f(x) \,\}$$

Use Theorem 2.10 page 47 to show that $G_f \in \mathcal{M} \otimes \mathcal{B}(R^+)$. Also, show that

$$(\mu \times m)(G_f) \;=\; \int_X f(x) \, d\mu \;=\; \int_0^\infty \mu\big(\{\, f > y \,\}\big)dy$$

(This shows that the integral can be viewed as " the area under the curve ".)

6. (Like Problem 51 page 69) Let $(X, \mathcal{M}_1, \mu)$ and $(Y, \mathcal{M}_2, \beta)$ be two arbitrary measure spaces, not necessarily σ-finite (so that you cannot apply Fubini's theorem directly). Show

(a) If $f : X \to R$ is $\mathcal{M}_1$-measurable and $g : Y \to R$ is $\mathcal{M}_2$-measurable for the real numbers R, then $h(x, y) = f(x)g(y)$ is $(\mathcal{M}_1 \otimes \mathcal{M}_2)$-measurable.

(b) If $f \in L^1(\mu)$ and $g \in L^1(\beta)$, then $h \in L^1(\mu \times \beta)$ and

$$\int h(z)d(\mu \times \beta)(z) = \Big(\int f(x)d\mu(x)\Big)\Big(\int g(y)d\beta(y)\Big)$$

7. (Like Problem 54 page 77) Theorem 2.44 has four statements about invertible $n \times n$ matrices T. Replacing " Lebesgue measurable function " by " Borel function " and $\mathcal{L}^n$ by $\mathcal{B}(R^n)$, which of the four statements are true or false if T is not invertible?

8. (Like Problem 61 page 77) Define

$$I_\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x - y)^{\alpha - 1} f(y)\, dy$$

for $\alpha > 0$, $x \in (0, \infty)$, and $f(x)$ continuous on $[0, \infty)$. The functional $I_\alpha f(x)$ is called the α^{th} *fractional integral* of $f(x)$. Show that

(a) $I_{\alpha+\beta} f(x) = I_\alpha\big(I_\beta f\big)(x)$ for all $\alpha, \beta > 0$ and $x > 0$. (You can use the identity in Problem 60 page 77 in the text.)

(b) If n is an integer with $n \geq 1$, then $(d/dx)^n I_n f(x) = f(x)$ for $x > 0$. (In this sense, $I_n f(x)$ is an n-fold integral of $f(x)$.)