Ma 5051 — Real Variables and Functional Analysis

Solutions for Take-Home Midterm

Prof. Sawyer — Washington University

Let $(X, \mathcal{M}, \mu)$ be a measure space. Recall $\int_A f(x)d\mu = \int I_A(x)f(x)d\mu$ for $A \in \mathcal{M}$ and $f \in L^+$, where $I_A(x)$ is the indicator function of A.

1. (a) The problem is to show that the integrand is dominated by an integrable function. Since $1 - e^{-x} = \int_0^x e^{-y}dy \le \int_0^x dy = x$, it follows that $1 - x \le e^{-x}$, $1 - (1/n)x \le e^{-x/n}$, and $\big(1 - (1/n)x\big)^n \le e^{-x}$. Thus the integrand is dominated by $x^k e^{-x}$, which is integrable on $(0, \infty)$. Then by the dominated convergence theorem

$$\begin{aligned}\lim_{n\to\infty} \int_0^n x^k \big(1 - n^{-1}x\big)^n dx &= \int_0^\infty \lim_{n\to\infty} I_{(0,n]}(x)\, x^k \big(1 - n^{-1}x\big)^n dx \\ &= \int_0^\infty x^k e^{-x}\, dx = k!\end{aligned}$$

(b) The substitution $x \to x/\sqrt{n}$ changes the integral to

$$I(n) = \int_0^{\sqrt{n}} \frac{1 + x^2}{\big(1 + (1/n)x^2\big)^n}\, dx$$

By the binomial theorem (see also the model solutions for Problem 4a of HW5),

$$\left(1 + \frac{x^2}{n}\right)^n \ge 1 + n\frac{x^2}{n} + \frac{n(n-1)}{2}\left(\frac{x^2}{n}\right)^2 \ge 1 + \frac{1}{3}x^4$$

for $x > 0$ and $n \ge 3$. Thus the integrand of $I(n)$ is dominated by $3(1+x^2)/(3+x^4)$, which is integrable on $(0, \infty)$. Then by the dominated convergence theorem

$$\begin{aligned}\lim_{n\to\infty} I(n) &= \int_0^\infty \lim_{n\to\infty} I_{(0,\sqrt{n}]}(x) \frac{1 + x^2}{\big(1 + (1/n)x^2\big)^n}\, dx \\ &= \int_0^\infty \frac{1 + x^2}{e^{x^2}}\, dx = \int_0^\infty e^{-x^2}\, dx + \int_0^\infty \frac{1}{2}x\left(2xe^{-x^2}\right) dx \\ &= \frac{3}{2}\int_0^\infty e^{-x^2}\, dx = \frac{3}{4}\sqrt{\pi}\end{aligned}$$

2. The integral is

$$I(a) = \int_0^\infty e^{-y^2} \sin(ay)\, dy = \int_0^\infty e^{-y^2} \sum_{n=0}^\infty \frac{(-1)^n a^{2n+1} y^{2n+1}}{(2n+1)!}\, dy$$

The partial sums of the integrand of $I(a)$ are dominated for fixed $a > 0$ by

$$e^{-y^2}\sum_{n=0}^{\infty}\frac{a^{2n+1}y^{2n+1}}{(2n+1)!} \;\le\; e^{-y^2}\sum_{n=0}^{\infty}\frac{a^n y^n}{n!} \;=\; e^{-y^2}e^{ay}$$

where the second series is formed from the first by adding in the even terms $a^{2n}y^{2n}/(2n)!$. Since the last expression above is integrable on $(0,\infty)$, dominated convergence allows us to interchange the integral and sum in $I(a)$ and conclude

$$\begin{aligned} I(a) &= \sum_{n=0}^{\infty}\frac{(-1)^n a^{2n+1}}{(2n+1)!}\int_0^{\infty} e^{-y^2}y^{2n+1}\,dy \;=\; \sum_{n=0}^{\infty}\frac{(-1)^n a^{2n+1}}{(2n+1)!}\,\frac{1}{2}\int_0^{\infty} e^{-x}x^n\,dx \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n a^{2n+1}\,n!}{2(2n+1)!} \;=\; \frac{a}{2}\sum_{n=0}^{\infty}\frac{(-a^2/2)^n}{(2n+1)(2n-1)\ldots(3)(1)} \end{aligned}$$

since $2^n n! = (2n)(2n-2)\ldots 2$.

One can use complex-variable techniques to show $I(a) = \int_0^{\infty} e^{-y^2}\sin(ay)\,dy = e^{-a^2/4}\int_0^{a/2} e^{x^2}\,dx$, but that is not required.

See the **Appendix** for two derivations of this identity using complex-variable methods, one based on analytic continuation and one using Cauchy's theorem.

3. Let $A = \liminf_{n\to\infty}\int f_n(x)\,d\mu$. Then there exists a sequence $n_k \uparrow \infty$ such that $\lim_{k\to\infty}\int f_{n_k}\,d\mu = A$. Since $f_{n_k} \to f$ in measure, there exists a further subsequence $\{\, f_{n_{k_j}} \}$ such that $\lim_{j\to\infty} f_{n_{k_j}}(x) = f(x)$ a.e. μ. Since $f_n(x) \ge 0$, this implies

$$\int f(x)\,d\mu \;\le\; \liminf_{j\to\infty}\int f_{n_{k_j}}(x)\,d\mu \;=\; \lim_{k\to\infty}\int f_{n_k}(x)\,d\mu \;=\; \liminf_{n\to\infty}\int f_n(x)\,d\mu$$

and $\int f(x)\,d\mu \le \liminf_{n\to\infty}\int f_n(x)\,d\mu$.

4. Since μ is a finite measure on $X = [0,1]$, by the dominated convergence theorem

$$\lim_{n\to\infty}\int_0^1\int_0^1 e^{-n(x-y)^2}d\mu(x)d\mu(y) \;=\; I \;=\; \int_0^1\int_0^1 J(x,y)d\mu(x)d\mu(y)$$

where $J(x,x) = 1$ and $J(x,y) = 0$ for $y \ne x$. The inner integral in I above, as a function of y, is

$$\int_0^1 J(x,y)d\mu(x) \;=\; \int_0^1 J(x,y)(dm(x) + 2d\delta_a(x)) \;=\; 2J(a,y)$$

since, for fixed y, $J(x,y) = 0$ (Lebesgue) a.e. Thus the double integral is

$$I \;=\; \int_0^1 2J(a,y)d\mu(y) \;=\; \int_0^1 2J(a,y)(dm(y) + 2d\delta_a(y)) \;=\; 4J(a,a) = 4$$

5. (a) Show that $G_f = \{\,(x,y) : 0 \le y < f(x)\,\}$ is measurable in the product σ-algebra (that is, $G_f \in \mathcal{M} \otimes \mathcal{B}(R^+)$).

Proof I of (a): By Proposition 2.10, there exist simple functions $f_n(x)$ such that $0 \le f_n(x) \uparrow f(x)$ for all x. If $h(x) = \sum_{j=1}^m c_j I_{A_j}(x)$ is a nonnegative simple function with disjoint $A_j \in \mathcal{M}$, then $G_h = \bigcup_{j=1}^m \big(A_j \times [0, c_j)\big)$ is a finite disjoint union of measurable rectangles and so is product measurable. Since $G_{f_n} \uparrow G_f$ if $0 \le f_n(x) \uparrow f(x)$ for all x, the set G_f is also product measurable.

Proof II of (a): The functions $F_1(x,y) = f(x)$ and $F_2(x,y) = y$ are both measurable in the product σ-algebra, since in each case $\{\,(x,y) : F_j(x,y) \le \lambda\,\}$ is a measurable rectangle. Thus $F(x,y) = F_1(x,y) - F_2(x,y) = f(x) - y$ is also product measurable, by Proposition 2.6 (page 45) in the text. Finally, $G_f = \{\,(x,y) : F(x,y) > 0\,\}$.

(b) By the Fubini-Tonelli theorem for the product measure of μ on $(X,\mathcal{M})$ and Lebesgue measure m on $R^+ = [0,\infty)$,

$$(\mu \times m)(G_f) \;=\; \int_X \int_0^\infty I_{G_f}(x,y)\,dm(y)d\mu(x) \;=\; \int_0^\infty \int_X I_{G_f}(x,y)\,d\mu(x)dm(y)$$

For fixed x, the section $(G_f)_x = \{\,y : y < f(x)\,\} = [0, f(x))$, so that the inner integral of the first integral above is $\int_0^\infty I_{G_f}(x,y)\,dm(y) = f(x)$. Thus the first iterated integral equals $\int_X f(x)\,d\mu$.

For fixed y, the section $(G_f)^y = \{\,x : f(x) > y\,\}$, so that $\int_X I_{G_f}(x,y)\,dm(y) = F(y) = \mu(\{\,x : f(x) > y\,\})$ and the second integral above is $\int_0^\infty F(y)\,dy$. Thus the two expressions are the same, and are also the same as $(\mu \times m)(G_f)$.

6. It is sufficient to assume $f(x) \ge 0$ and $g(y) \ge 0$, since otherwise we can write $f = f^+ - f^-$ and $g = g^+ - g^-$. Define simple functions $f_n(x), g_m(y)$ such that $0 \le f_n(x) \uparrow f(x)$ for all x and $0 \le g_m(y) \uparrow g(y)$ for all y.

(a) If $f_n(x) = \sum_{j=1}^{m_1} a_j I_{A_j}(x)$ and $g_n(y) = \sum_{k=1}^{m_2} b_j I_{B_j}(y)$ for fixed n and disjoint sets $\{\,A_j\,\}$ in X and $\{\,B_k\,\}$ in Y, then $\{\,A_j \times B_k\,\}$ are disjoint subsets of $X \times Y$ and

$$\{\,(x,y) : f_n(x)g_n(y) \le \lambda\,\} \;=\; \bigcup\bigcup \{\,A_j \times B_k : a_j b_k \le \lambda\,\}$$

is a finite union of measurable rectangles for each λ. Thus the functions $f_n(x)g_n(y)$ are product measurable.

Since $0 \le f_n(x)g_n(y) \uparrow h(x,y) = f(x)g(y)$ for all x and y, it follows that $h(x,y)$ is product measurable. (You can also argue directly from Proposition 2.6 as in Proof II of Problem 5a.)

(b) **Proof I**. Since $f \in L^1(\mu)$ and $g \in L^1(\beta)$, the sets $X_1 = \{\, x : f(x) > 0 \,\}$ and $Y_1 = \{\, y : g(y) > 0 \,\}$ are σ-finite. The function $h(x,y) = 0$ for $(x,y) \notin X_1 \times Y_1$. By the definition of integrals as the supremum of simple functions underneath, $\int_{X\times Y} h(x,y)d\nu = \int_{X_1\times Y_1} h(x,y)d\nu$. Thus part (b) follows from Tonelli's theorem restricted to $X_1 \times Y_1$.

Proof II. By definition, $(\mu\times\beta)(A\times B) = \mu(A)\beta(B)$ for $A \in \mathcal{M}_1$ and $B \in \mathcal{M}_2$. If $f_1(x) = \sum_{j=1}^n c_j I_{A_j}(x)$ for disjoint $A_j \in \mathcal{M}_1$ and $g_1(y) = \sum_{k=1}^m d_k I_{B_k}(y)$ for disjoint $B_k \in \mathcal{M}_2$, then

$$\int f_1(x)g_1(y)\, d(\mu\times\beta) \;=\; \sum_{j=1}^n \sum_{k=1}^m c_j d_k\, (\mu\times\beta)(A_j\times B_k)$$

$$= \left(\sum_{j=1}^n c_j \mu(A_j)\right)\left(\sum_{k=1}^m d_k \beta(B_k)\right) \;=\; \left(\int f_1(x)\, d\mu\right)\left(\int g_1(y)\, d\beta\right)$$

Thus the identity is true for nonnegative simple functions. Since $0 \le f_n(x)g_n(y) \uparrow h(x,y) = f(x,y)$ for all x and y, it follows from the increasing limits theorem that it is true for $h(x,y)$ as well.

7. (a) "If f is Borel measurable on R^n, then so is $f\circ T$" This is true even if T is not invertible, since, since by the definition of a measurable function on page 43, a Borel function of a Borel-measurable function is always Borel measurable.

(b) (Equation (2.45)) If T is not invertible and (for example) $f(x) = \exp(-\|x\|^2)$, then $\det(T) = 0$ and $\int f\big(T(x)\big)\, dx = \infty$. The equation is false unless you have a very specific convention about the product $0\times\infty$.

(c) "If $E \in \mathcal{B}(R^n)$, then $T(E) \in \mathcal{B}(R^n)$" This is true if T is invertible since both T and T^{-1} are measurable. For non-invertible T, it is equivalent to whether or not an orthogonal projection of a Borel set onto a lower-dimensional subspace is always a Borel set. Unfortunately, this is apparently false even for $n = 2$ by a theorem of Suslin, but Lebesgue apparently gave a false proof of this result. Thus you are in good company if you believed that it was true. (Part (c) not counted for grade on Problem 7.)

(d) "$m(T(E)) = |\det(T)|m(E)$" is true, in a sense, if T is not invertible, since (i) the range of T is a lower-dimensional subspace of R^n and (ii) $\det(T) = 0$. Thus $T(E)$ is a Lebesgue null set even if it is not a Borel set.

8. (a) If $f(x)$ is continuous on $[0,\infty)$ and $\alpha > 0$, then by dominated convergence

$g(x) = I_\alpha f(x)$ is also continuous on $[0,\infty)$ with $g(0)=0$. If $x>0$,

$$\begin{aligned}
I_\alpha\big(I_\beta f\big)(x) &= \frac{1}{\Gamma(\alpha)}\int_0^x (x-y)^{\alpha-1} I_\beta(f)(y)dy \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^x (x-y)^{\alpha-1}\int_0^y (y-z)^{\beta-1} f(z)dzdy \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^x\int_0^x I_{\{\,(y,z):0\le z\le y\le x\,\}}(y,z)(x-y)^{\alpha-1}(y-z)^{\beta-1} f(z)dzdy \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^x f(z)\int_z^x (x-y)^{\alpha-1}(y-z)^{\beta-1} dy\,dz \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^x f(z)\int_0^{x-z} (x-z-y)^{\alpha-1} y^{\beta-1} dy\,dz \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^x f(z)(x-z)^{\alpha+\beta-1}\int_0^1 (1-y)^{\alpha-1} y^{\beta-1} dy\,dz \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\int_0^x f(z)(x-z)^{\alpha+\beta-1}dz \;=\; I_{\alpha+\beta}f(x)
\end{aligned}$$

by the identity $\int_0^1 x^{a-1}(1-x)^{b-1}dx = \Gamma(a)\Gamma(b)/\Gamma(a+b)$ (see Problem 60 on page 77).

(b) If $n=1$, $I_1 f(x) = \int_0^x f(y)dy$ and $(d/dx)I_1 f(x) = f(x)$. If $n>1$, then $I_n f(x) = I_1(I_{n-1}f)(x) = \int_0^x I_{n-1}f(y)dy$ by part (a) and $(d/dx)I_n f(x) = I_{n-1}f(x)$. Thus $(d/dx)^n I_n f(x) = f(x)$ by induction.

Appendix: We now show how to derive

$$I(a) \;=\; \int_0^\infty e^{-y^2}\sin(ay)\,dy \;=\; e^{-a^2/4}\int_0^{a/2} e^{x^2}\,dx \qquad (*)$$

by using *either* analytic continuation *or* Cauchy's theorem. Both cases depend on the useful identity $e^{iay} = \cos(ay) + i\,\sin(ay)$.

By Analytic Continuation: By completing a square,

$$\int_0^\infty e^{-y^2}e^{ay}\,dy \;=\; \int_0^\infty e^{-(y-a/2)^2}e^{a^2/4}\,dy \;=\; \left(\int_{-a/2}^\infty e^{-y^2}\,dy\right)e^{a^2/4}$$

and thus

$$\int_0^\infty e^{-y^2}\left(\frac{e^{ay}-e^{-ay}}{2}\right)dy \;=\; e^{a^2/4}\left(\frac{1}{2}\int_{-a/2}^{a/2} e^{-y^2}\,dy\right) \;=\; e^{a^2/4}\int_0^{a/2} e^{-y^2}\,dy \qquad (**)$$

In general, if two functions

$$f_1(a) = \sum_{j=0}^{\infty} b_n a^n \qquad \text{and} \qquad f_2(a) = \sum_{j=0}^{\infty} c_n a^n$$

(i) equal power-series expansions that converge for all complex a and (ii) satisfy $f_1(a) = f_2(a)$ for all real $a > 0$, then $f_1(a) = f_2(a)$ for all complex a. This is because $b_n = c_n = ((d^n/da^n)f_1)(0)/n!$ where the derivatives can be calculated for $a > 0$ only, so that the power series are identical. (In fact, it is only necessary that $f_1(a) = f_2(a)$ on a sequence $a_n \to 0$.) Thus, using $\sin(ay) = (e^{iay} - e^{-iay})/(2i)$, replacing a by ia in (**) yields

$$\int_0^{\infty} e^{-y^2} \sin(ay)\, dy \;=\; i\, e^{(ia)^2/4} \int_0^{ia/2} e^{-y^2}\, dy \;=\; i\, e^{-a^2/4} \int_0^{ia/2} e^{-y^2}\, dy$$

(In this case, we say that we have *analytically continued* (**) from real a to $z = ia/2$.) The last integral above may look odd if you are not used to integrals in the complex plane, but stands for a path integral along any path $y = y(t)$ for $0 \le t \le 1$ with $y(0) = 0$ and $y(1) = ia/2$. In particular we can use the substitution $y \to iy$ in the integral to obtain (*).

By Cauchy's Theorem: Cauchy's theorem implies that $\int_C f(z)dz = 0$ for any closed path C and function $f(z)$ that is analytic in the plane (that is, equals a power-series expansion that converges for all z). By completing a square in the exponent

$$J(a) \;=\; \int_0^{\infty} e^{-y^2} e^{iay}\, dy \;=\; e^{-a^2/4} \int_0^{\infty} e^{-(y-ia/2)^2}\, dy$$

Consider a closed path C consisting of four line segments: $I_1(T)$ is a straight line from $-ia/2$ to $(T - ia/2)$ oriented from left to right, $I_2(T)$ from $(T - ia/2)$ to T, $I_3(T)$ from T to 0 oriented from right to left, and $I_4(T)$ from 0 to $-ia/2$. By Cauchy's theorem, the sum of the integrals of $f(z) = e^{-z^2}$ over the four paths is equal to zero. As $T \to \infty$, the integrals over $I_1(T) \to I_1(\infty)$, $I_2(T) \to 0$, and $I_3(T) \to I_3(\infty)$. Thus the sum of the integrals of $f(z)$ over $I_1(\infty)$, $I_3(\infty)$ (oriented from right to left), and I_4 is equal to zero. This implies

$$J(a) \;=\; e^{-a^2/4} I_1(\infty) \;=\; -e^{-a^2/4} I_3(\infty) \;-\; e^{-a^2/4} I_4$$

Now $-I_3(\infty) = \int_0^{\infty} e^{-x^2}\, dx = \sqrt{\pi}/2$ and

$$-I_4 \;=\; -\int_0^{-ia/2} e^{-z^2}\, dz \;=\; i \int_0^{a/2} e^{z^2}\, dz$$

by making the substitution $z \to -iz$. This implies

$$J(a) \;=\; \int_0^\infty e^{-y^2} e^{iay}\, dy \;=\; e^{-a^2/4} \left(\frac{\sqrt{\pi}}{2} \;+\; i \int_0^{a/2} e^{x^2}\, dx \right)$$

Taking real and imaginary parts in the equation above leads to

$$\int_0^\infty e^{-y^2} \cos(ay)\, dy \;=\; e^{-a^2/4} \frac{\sqrt{\pi}}{2}$$

$$\int_0^\infty e^{-y^2} \sin(ay)\, dy \;=\; e^{-a^2/4} \int_0^{a/2} e^{x^2}\, dx$$