Ma 551 — Advanced Probability

Problem Set #1 — Due October 2, 2007

Prof. Sawyer — Washington University

Text references are to Kai Lai Chung, *A Course in Probability Theory*, 3rd edition, Academic Press, 2001.

1. Let $\Omega = \{0, 1, 2, \ldots\}$ be the nonnegative integers and let $\mathcal{A}$ be the set of all finite subsets of Ω. Show that

(i) $\mathcal{A}$ is a semi-ring of subsets of Ω.

(ii) $\mathcal{B}(\mathcal{A}) = 2^{\Omega}$, where $\mathcal{B}(\mathcal{A})$ is the smallest σ-algebra of subsets of Ω that contains $\mathcal{A}$ and 2^{Ω} means the set of all subsets of Ω.

(iii) There does not exist a probability measure $(\Omega, \mathcal{F}, P)$ for $\mathcal{F} = \mathcal{B}(\mathcal{A})$ such that $\mu(\{n\}) = \alpha$ has the same value for all $n \in \Omega$.

2. Let $X \geq 0$ be a nonnegative random variable (r.v.) on a probability space $(\Omega, \mathcal{F}, P)$ and assume that $r > 0$. Show that

$$E(X^r) = \int_0^\infty r y^{r-1} P(X \geq y)\, dy$$

Conclude that $E(X) < \infty$ if and only if

$$\int_0^\infty P(X \geq y)\, dy < \infty$$

(*Hint*: Write $E(X^r) = E\big(\int_0^{X^r} du\big)$, express in terms of a product measure $\mu_2(d\omega dy)$ on $\Omega \times [0, \infty)$, and use Fubini's Theorem.)

3. Let X be a uniformly distributed r.v. on a probability space $(\Omega, \mathcal{F}, P)$. (That is, $P(X \leq x) = x$ for $0 \leq x \leq 1$.) Let $Y = r \log(1/X)$ for some $r > 0$. Prove that Y has a density function $f_Y(y)$ with respect to Lebesgue measure and find $f_Y(y)$. (*Hint*: This holds if and only if $E\big(\phi(Y)\big) = \int_0^\infty \phi(y) f_Y(y) dy$ for all bounded Borel functions $\phi(x) \geq 0$.)

4. Let $\{ X_n \}$ be an independent and identically distributed sequence of random variables on a probability space $(\Omega, \mathcal{F}, P)$ such that $X_i \geq 0$ and $E(X_i) = \infty$. Use the strong law of large numbers for independent random variables with $E(X^4) < \infty$ to prove that

$$\lim_{n\to\infty} \frac{X_1 + X_2 + \cdots + X_n}{n} = \infty \quad \text{a.s.}$$

(*Hint*: Consider $X_i^C = \min\{X_i, C\}$ for $C > 0$.)

5. Let X and Y be independent and identically distributed random variables. Find the limit

$$\lim_{n\to\infty} \operatorname{Exp}\left(e^{-n(X-Y)^2}\right)$$

(*Hint*: Express the expectation in terms of the joint distribution function of X and Y. If you take a limit inside an integral, explain why it is valid. Be careful!)

6. Let $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$ be two sets of independent random variables on a probability space $(\Omega, \mathcal{F}, P)$. Assume that the X_i have a density $f(x) = f_X(x)$ on R^1 and the Y_j have density $g(y) = f_Y(y)$, respectively. That is,

$$P(X_i \le x) = F_X(x) = \int_0^x f(u)du, \qquad \text{all } i, \text{ all } x$$
$$P(Y_j \le y) = F_Y(y) = \int_0^y g(v)dv, \qquad \text{all } j, \text{ all } y$$

Assume $g(y) > 0$ for all $y \in R^1$ and define

$$L_R(y_1, y_2, \ldots, y_n) = \frac{f(y_1)f(y_2)\ldots f(y_n)}{g(y_1)g(y_2)\ldots g(y_n)}$$

Prove that

$$E\Big(\phi(Y_1, Y_2, \ldots, Y_n)L_R(Y_1, \ldots, Y_n)\Big) = E\Big(\phi(X_1, X_2, \ldots, X_n)\Big)$$

for all nonnegative bounded continuous functions $\phi(y_1, \ldots, y_n)$ on R^n. (*Hint*: Use the fact that X_i and Y_j are independent and write out both sides as integrals on R^n.)