Ma 551 — Advanced Probability

Problem Set #3 — Due November 20, 2007

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Text references are to Kai Lai Chung, *A Course in Probability Theory*, 3rd edition, Academic Press, 2001.

1. Let $\{ X_n, \mathcal{F}_n \}$ be a submartingale on a probability space $(\Omega, \mathcal{F}, P)$ such that $E(X_n) = E(X_1)$ for all $n \geq 1$. Prove that $\{ X_n, \mathcal{F}_n \}$ is a martingale.

2. Let X_i be i.i.d. such that $\phi(\theta) = \mathrm{Exp}\left(e^{\theta X_i}\right) < \infty$ for all real θ and $E(X_i) < 0$. Let $S_n = X_1 + X_2 + \ldots + X_n$. Then $\lim_{n\to\infty} S_n = -\infty$ a.s. by the strong law of large numbers, but, if $P(X_i > 0) > 0$, we can think about the largest positive value that S_n takes on before converging to $-\infty$. Prove that there exists a value $\theta_0 > 0$ such that for all $\lambda > 0$,

$$\Pr\left(\max_{1\leq n<\infty} S_n \geq \lambda\right) \;\leq\; e^{-\theta_0 \lambda}$$

(*Hints*: Show $\phi(0) = 1$, $\phi'(0) = E(X_i) < 0$, and $\lim_{\theta\to\infty} \phi(\theta) = \infty$. Conclude that $\phi(\theta_0) = 1$ for some $\theta_0 > 0$. Show that $E\left(e^{\theta S_n}\right) = \phi(\theta)^n$ and conclude that $X_n(\theta) = \exp(\theta S_n)\phi(\theta)^{-n}$ is a martingale.)

3. Let $\{ X_n, \mathcal{F}_n \}$ be a martingale with uniformly bounded increments. (That is, $|X_n(\omega) - X_{n-1}(\omega)| \leq C$ for all ω for some C.) Prove that, with probability one,

$$\limsup_{n\to\infty} X_n(\omega) = +\infty \tag{4.1}$$

implies

$$\liminf_{n\to\infty} X_n(\omega) = -\infty \tag{4.2}$$

(*Hint*: Use Doob's Theorem for nonnegative martingales and the Optional Stopping Theorem to show that $\sup_n X_n(\omega) = \infty$ implies $\inf_n X_n(\omega) = -\infty$ except for a set of ω with probability zero. This type of argument is sometimes called "stopping before you get into trouble", which can be good advice even if you are not a stochastic process.)

4. Suppose that $p_{ij} \geq 0$ for integers $-\infty < i, j < \infty$ with $\sum_k p_{ik} = 1$ and $p_{ii} < 1$ for all i. Define random variables $X_n \in J = \{ \ldots, -m, \ldots, 0, \ldots, n, \ldots \}$ corresponding to $X_0 = 0$ a.s. and p_{ij} as in Problems 3 and 4 in Problem Set #2. Equivalently,

X_n is the Markov chain generated by p_{ij} with $X_0 = 0$. Say that a function $\phi(k)$ on J is *p-harmonic* if $\phi(i) = \sum_{k \in J} p(i,k)\phi(k)$ for all $i \in J$.

Suppose that there exists a nonnegative p-harmonic function $\phi(i)$ on J that is strictly increasing and unbounded on J. (That is, such that $\phi(n) < \phi(n+1)$ for all n and $\sup_n \phi(n) = \infty$.) Prove that

$$\lim_{n \to \infty} X_n(\omega) = -\infty \qquad \text{a.s.}$$

This shows that there is a connection between the harmonic functions of an infinite matrix p_{ij} and the sample behavior of X_n. (*Hint*: Is $\{\, \phi(X_n), \mathcal{F}_n \,\}$ a martingale for $\mathcal{F}_n = \mathcal{B}(X_1, \ldots, X_n)$? A submartingale?)