Ma 551 — Advanced Probability

Problem Set #1 — Due October 6, 2009

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Problems in text are from Patrick Billingsley, *Probability and Measure*, 3rd edn, John Wiley & Sons, 1995.

1. Let $X \geq 0$ be a nonnegative random variable (r.v.) on a probability space $(\Omega, \mathcal{F}, P)$ and assume that $r > 0$. Show that

$$E(X^r) = \int_0^\infty r y^{r-1} P(X \geq y)\, dy$$

Conclude that $E(X) < \infty$ if and only if

$$\int_0^\infty P(X \geq y)\, dy \ < \ \infty$$

(*Hint*: Write $E(X^r) = E\big(\int_0^{X^r} du\big)$, express in terms of a product measure $\mu_2(d\omega dy)$ on $\Omega \times [0, \infty)$, and use Fubini's Theorem.)

2. Let X be a uniformly distributed r.v. on a probability space $(\Omega, \mathcal{F}, P)$. (That is, $P(X \leq x) = x$ for $0 \leq x \leq 1$.)

(i) Show that $E\big(\phi(X)\big) = \int_0^1 \phi(x) dx$ for all Borel functions $\phi(x) \geq 0$. (*Hint*: Use the first lifting theorem.)

(ii) Let $Y = r \log(1/X)$ for some $r > 0$. Prove that Y has a density function $f_Y(y)$ with respect to Lebesgue measure and find $f_Y(y)$. (*Hint*: This holds if and only if $E\big(\phi(Y)\big) = \int_0^\infty \phi(y) \mu_{F_Y}(dy) = \int_0^\infty \phi(y) f_Y(y) dy$ for all bounded Borel functions $\phi(x) \geq 0$.)

3. Let $\{ X_n \}$ be an independent and identically distributed sequence of random variables on a probability space $(\Omega, \mathcal{F}, P)$ such that $X_n \geq 0$ and $E(X_n) = \infty$. Use the strong law of large numbers for independent random variables with $E(X^4) < \infty$ to prove that

$$\lim_{n \to \infty} \frac{X_1 + X_2 + \cdots + X_n}{n} \ = \ \infty \quad \text{a.s.}$$

(*Hint*: Consider $X_i^C = \min\{X_i, C\}$ for $C > 0$.)

4. Let X and Y be independent and identically distributed random variables. Find the limit

$$\lim_{n\to\infty} E\big(e^{-n(X-Y)^2}\big)$$

(*Hint*: Express the expectation in terms of the joint distribution function of X and Y. If you take a limit inside an integral, explain why it is valid. What happens if the distributions of X and Y have atoms? Be careful!)

5. Let X and Y be random variables on a probability space $(\Omega, \mathcal{F}, P)$. Assume that X has density $f(x) = f_X(x)$ on $R = R^1$ and Y has density $g(y) = f_Y(y)$. That is,

$$P(X \le x) = F_X(x) = \int_0^x f(u)du, \qquad \text{all } x$$
$$P(Y \le y) = F_Y(y) = \int_0^y g(v)dv, \qquad \text{all } y$$

Assume $g(y) > 0$ for all $y \in R^1$ and define

$$L_R(y) \;=\; \frac{f(y)}{g(y)}$$

Prove that

$$E\Big(\phi(Y)L_R(Y)\Big) \;=\; E\Big(\phi(X)\Big)$$

for all nonnegative Borel functions $\phi(y)$ on R. (*Hint*: Use the first lifting theorem.)