Ma 551 — Advanced Probability

Problem Set #2 — Due October 20, 2009

Prof. Sawyer — Washington University

Problems in text are from Patrick Billingsley, *Probability and Measure*, 3rd edn, John Wiley & Sons, 1995. The abbreviation i.r.v means independent random variables and i.i.d means independent and identically distributed (random variables).

Six problems on two pages.

1. (Like Problem 21.14 page 281) Assume $E(|X+Y|)<\infty$ for i.r.v. X,Y. Show that $E(|X|)<\infty$ and $E(|Y|)<\infty$. Also show by counterexample that the conclusion may be false if X,Y are not independent.

2. Let $X_1,X_2,\ldots,X_n,\ldots$ be i.i.d. Show that $X_n=O(n)$ a.s. if and only if $E(|X_1|)<\infty$. (*Hint*: $X(n)=O(n)$ a.s. means that, with probability one, there exists a constant $C(\omega)<\infty$ such that $|X_n(\omega)|\le C(\omega)n$ for $n\ge n_0$, where $n_0<\infty$ can also depend on ω. Consider Problem 1 of HW1 with $r=1$.)

3. (Problem 22.1 page 294) Let $X_1,X_2,\ldots,X_n,\ldots$ be i.r.v. Let Y be a random variable that is $\mathcal{T}$-measurable where $\mathcal{T}$ is the σ-algebra

$$\mathcal{T}=\bigcap_{n=1}^{\infty}\mathcal{B}(X_n,X_{n+1},\ldots,X_{n+m},\ldots)$$

Prove that there exists a constant a such that $Y=a$ a.s.

4. (Problem 22.7 page 295) Let $X_1,X_2,\ldots,X_n,\ldots$ be i.i.d. with $E(|X_1|)=\infty$. Conclude that $\sup_n\frac{|X_n|}{n}=\infty$ a.s. Use this to show

$$\limsup_{n\to\infty}\left|\frac{X_1+X_2+\ldots+X_n}{n}\right| = \infty \text{ a.s.}$$

(*Hint*: If $S_n=X_1+\ldots+X_n$, find a relation between S_n/n, X_n/n, and $S_{n-1}/(n-1)$.)

5. (Like Problem 22.8 page 295) Let $X_1,X_2,\ldots$ be i.i.d. with $E(|X_i|)<\infty$ and $E(X_i)=\mu$. A *stopping time* for $X_1,X_2,\ldots$ is an integer valued random variable $\tau\ge 1$ such that $\{\,\tau=n\,\}\in\mathcal{B}(X_1,\ldots,X_n)$ for all $1\le n<\infty$. Set $S_n=X_1+X_2+\ldots+X_n$.

(a) Let

$$\tau_1 = \min\{\, n : \sum_{j=1}^{n} X_j^2 \geq 4 \,\} \quad \text{and} \quad \tau_2 = \min\{\, n : |S_n| \geq 17 \,\}$$

with the convention $\min\{\,\phi\,\} = \infty$, so that $\tau_1 = \infty$ or $\tau_2 = \infty$ if the event never occurs. Show that τ_1 and τ_2 are stopping times.

(b) Prove *Wald's Lemma*: If τ is any stopping time with $E(\tau) < \infty$, then

$$E(S_\tau) = E(\tau)E(X_1)$$

where $S_\tau(\omega) = \sum_{j=1}^{\tau(\omega)} X_j(\omega)$. (*Hint*: If τ is a stopping time, then $\{\,\tau \leq n\,\} = \bigcup_{j=1}^{n}\{\,\tau = j\,\} \in \mathcal{B}(X_1, \ldots, X_n)$ but $\{\,\tau \geq n\,\} = \{\,\tau \leq n-1\,\}^c \in \mathcal{B}(X_1, \ldots, X_{n-1})$.)

(c) Suppose that each X_n is ± 1 with $P(X_n = 1) = p$ and $P(X_n = -1) = q$ for $p + q = 1$. For $p > q$, let τ be the first n such that either $S_n \leq -A$ (viewed as losing what you initially were willing to risk in a favorable game) or $S_n \geq B$ (viewed as breaking the bank) for integers $A, B \geq 1$.

Show that $E(\tau) < \infty$. Use Wald's lemma and the estimate $S_\tau \leq B$ to give an upper bound of the expected length of the game $E(\tau)$. (*Hints*: (i) If τ is a stopping time, show that $\tau_n = \min\{\,\tau, n\,\}$ is also a stopping time. (ii) This is the classical Gambler's Ruin problem. See Section 7 pages 92–94 for this problem and the rest of Section 7 for a series of increasing desperate strategies to make greater profits. Formulas in Section 7 would allow you to find $E(S_\tau)$ exactly, but that is not needed here.)

6. (Problem 23.10 page 310) (a) Let $X_1, X_2, \ldots, X_m, \ldots$ be i.i.d. with $X_i > 0$ a.s. and $E(X_i) = \mu < \infty$. Thus $S_n/n \to \mu$ a.s. Define

$$N_t(\omega) = \sup\{\, n : S_n(\omega) \leq t \,\}$$

as in equation (23.5) page 298. (If X_i represent the times to failure for a succession of light bulbs that are immediately replaced when they burn out, then N_t is the number of light bulbs that have burned out by time t.) Prove that $\lim_{t\to\infty} N_t/t = 1/\mu$ a.s.

(b) Let $X_1, X_2, \ldots$ be i.i.d. with $X_i > 0$ a.s. and $E(X_i) = \infty$. Prove that $\lim_{t\to\infty} N_t/t = 0$ a.s.