Ma 551 — Advanced Probability

Problem Set #3 — Due November 17, 2009

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Problems in text are from Patrick Billingsley, *Probability and Measure*, 3rd edn, John Wiley & Sons, 1995. Here i.r.v. means independent random variables and i.i.d. means independent and identically distributed (random variables).

Six problems on two pages.

1. Let $X_1, X_2, \ldots$ be r.v.s with $F_n(y) = P(X_n \le y)$. Assume $\sup_n E\big(h(X_n)\big) = C < \infty$ where $h(y) \ge 0$ and $\lim_{y \to \pm\infty} h(y) = \infty$. Prove that the family $\{\, F_n(y) \,\}$ is tight.

2. Let X_n, Y_n be random variables such that
(a) $P(X_n \le y) \to F(y)$ at all points of continuity of the d.f. $F(y)$
(b) The family $\{\, G_n(y) \,\}$ for $G_n(y) = P(Y_n \le y)$ is tight.
Assume $h_n \to 0$ as $n \to \infty$. Then show

$$P(X_n + h_n Y_n \le y) \to F(y)$$

at all points of continuity y of $F(y)$.

3. Let $M, X_1, X_2, \ldots, X_n, \ldots$ be i.r.v.s such that $P(X_k \le y) = F_X(y)$ for all y and k. (That is, the X_k are i.d.) Assume that M has the Poisson distribution $P(M = n) = e^{-\mu}\mu^n/n!$ for $n = 0, 1, 2, \ldots$ and let $Y = \sum_{j=1}^{M} X_j$. (That is, Y is the sum of a random number of i.i.d. with distribution $F_X(y)$.) Prove that

$$E\left(e^{i\theta Y}\right) \;=\; \exp\left[\mu \int \left(e^{i\theta y} - 1\right) F_X(dy)\right]$$

4. (Problem 14.5, p198) Define

$$\rho(F, G) = \inf\{\, \epsilon > 0 : \text{ for all } x,\ F(x - \epsilon) - \epsilon \le G(x) \le F(x + \epsilon) + \epsilon \,\}$$

for d.f.s F, G. Prove that
(a) ρ is a metric on d.f.s
(b) For d.f.s F_n, F, $F_n(y) \to F(y)$ weakly if and only if $\rho(F_n, F) \to 0$.

(For distributions on R, $\rho(F, G)$ is called the *Lévy distance* between F and G. The analog of ρ for distributions on a Banach space is called the *Skorokhod metric*. Among other things, this shows that the topology of weak convergence of d.f.s on R is a metric topology.)

5. (Like Problem 27.11, p368) Let $X_1, X_2, \ldots, X_n, \ldots$ be i.i.d. with density $f(x) = 1/|x|^3$ for $|x| \geq 1$ and $f(x) = 0$ for $|x| < 1$. Prove that

(a) $E\big(|X_k|^{2-\delta}\big) < \infty$ for all $\delta > 0$, $E(X_k) = 0$, but $E(X_k^2) = \infty$.

(b) Find a constant $c > 0$ such that for all real y

$$\lim_{n\to\infty} P\left(\frac{X_1 + X_2 + \cdots + X_n}{c\sqrt{n \log n}} \;\leq\; y\right) \;=\; \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-(1/2)x^2}\, dx$$

(*Hint*: Let $\varphi_n(\theta) = E\big(e^{i\theta S_n}\big)$ for $S_n = (X_1 + \cdots + X_n)/a_n$ for appropriate a_n. Since $\varphi(\theta) = E\big(e^{i\theta X_k}\big)$ is real and $|1 - \varphi(\theta)| < 1/2$ for $|\theta| < \delta$, you can take logarithms with a clear conscience. This gives an example of a central limit theorem for summands of infinite variance but with a larger denominator.)

6. Let X_k be i.r.v. with

$$\begin{aligned} P(X_k = +\sqrt{k}) &= 1/(2k) \\ P(X_k = 0) &= 1 - (1/k) \\ P(X_k = -\sqrt{k}) &= 1/(2k) \end{aligned}$$

for $k = 1, 2, \ldots$. Note that $E(X_k) = 0$ and $E(X_k^2) = 1$ for all k, but that X_k are not i.i.d. Let $S_n = (X_1 + \cdots + X_n)/\sqrt{n}$. Prove that

(a) For all θ,

$$\begin{aligned} \varphi_n(\theta) \;=\; E\big(e^{i\theta S_n}\big) \;&\to\; \exp\left(2\int_0^1 \frac{\cos(\theta y) - 1}{y}\, dy\right) \\ &= \exp\left(\int_{-1}^{1} \big(e^{i\theta y} - 1\big) \frac{dy}{|y|}\right) \end{aligned}$$

(b) For all points of continuity y of some d.f. $F(y)$

$$P\left(\frac{X_1 + \cdots + X_n}{\sqrt{n}} \;\leq\; y\right) \;\to\; F(y)$$

but $F(y)$ is not normal. (In particular, $F(y)$ is not the d.f. of $\mu + \sigma Z$ for any constants μ, σ where $Z \approx N(0, 1)$.)

Note the similarity between the characteristic function above and the characteristic function in Problem 3, although here the analog of $F_X(dy)$ is not normalizable.