Ma 551 — Advanced Probability

Solutions for Problem Set #3 due November 17, 2009

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Six problems. See the Math 551 Web site for the statement of problems.

1. For each $\epsilon > 0$, there exists $K = K_\epsilon$ such that $|h(y)| \geq 2C/\epsilon$ for $|y| \geq K$. Then $P(|X_n| \geq K) \leq P\big(|h(X_n)| \geq 2C/\epsilon\big) \leq (\epsilon/(2C))E(|h(X_n)|) < \epsilon$ for all $n \geq 1$, or equivalently $P\big(X_n \in [-K, K]\big) \geq 1 - \epsilon$ for all n.

2. Let $H_n(y) = P(X_n + h_n Y_n \leq y)$. For each $\epsilon > 0$, there exists $K = K_\epsilon$ such that $P(|Y_n| \geq K) < \epsilon$. Chose n_0 such that $|h_n| < \epsilon/K$ for $n \geq n_0$. Then, if $n \geq n_0$, $P(|h_n Y_n| > \epsilon) = P(|Y_n| > \epsilon/h_n) \leq P(|Y_n| > K) < \epsilon$. This implies

$$\begin{aligned} H_n(y) &= P(X_n \leq y - h_n Y_n) \leq P(X_n \leq y + \epsilon) + \epsilon \\ &\geq P(X_n \leq y - \epsilon) - \epsilon \end{aligned}$$

(***Remark***: This implies $\rho(H_n, F_n) \to 0$ for the Lévy metric $\rho(F, G)$ defined in Problem 4. Since $\rho(F_n, F) \to 0$ and ρ satisfies the triangle inequality, $\rho(H_n, F) \to 0$ and we are done. The rest of the proof is for those who have not yet done Problem 4.)

Assume that y and $y \pm \epsilon$ are points of continuity of $F(y)$. Then $P(X_n \leq y \pm \epsilon) \to F(y \pm \epsilon)$ and

$$F(y - \epsilon) - \epsilon \leq \liminf_{n\to\infty} H_n(y) \leq \limsup_{n\to\infty} H_n(y) \leq F(y + \epsilon) + \epsilon$$

Since y is a point of continuity of $F(y)$, and since the above holds for a sequence $\epsilon = \epsilon_m \to 0$, it follows that $H_n(y) \to F(y)$.

3. Let $\phi(\theta) = E\big(e^{i\theta X_1}\big) = \int_{-\infty}^{\infty} e^{i\theta y} F_X(dy)$. Then

$$E\left(e^{i\theta Y}\right) = \sum_{n=0}^{\infty} E\left(e^{i\theta Y} I_{[M=n]}\right) = \sum_{n=0}^{\infty} E\left(\prod_{k=1}^{n} e^{i\theta X_k} I_{[M=n]}\right)$$

Since the $\{\, M, X_1, \dots \}$ are independent,

$$E\left(e^{i\theta Y}\right) = \sum_{n=0}^{\infty} \phi(\theta)^n P(M = n) = e^{-\mu} \sum_{n=0}^{\infty} \mu^n \phi(\theta)^n / n! = \exp\big(\mu(\phi(\theta) - 1)\big)$$

which implies the desired result.

4. (a) (i) $\rho(F,F)=0$, and, since $F(x)$ and $G(x)$ are right continuous, $\rho(F,G)\neq 0$ if $F\neq G$. (ii) Note $F(x-\epsilon)-\epsilon\le G(x)$ and $G(x)\le F(x+\epsilon)+\epsilon$ for all x implies $G(x-\epsilon)-\epsilon\le F(x)$ and $F(x)\le G(x+\epsilon)+\epsilon$ for all x. Thus $\rho(F,G)=\rho(G,F)$. (iii) Suppose that $\rho(F,G)<\epsilon_1$ and $\rho(G,H)<\epsilon_2$. Then, for all x,

$$H(x-\epsilon_1-\epsilon_2)-\epsilon_1-\epsilon_2 \;\le\; G(x-\epsilon_1)-\epsilon_1 \;\le\; F(x)$$

with similar upper inequalities. Thus $\rho(F,H)<\epsilon_1+\epsilon_2$ and ρ satisfies the triangle inequality.

(b) If $\rho(F_n,F)\to 0$, then

$$F(y-\epsilon)-\epsilon \;\le\; F_n(y) \;\le\; F(y+\epsilon)+\epsilon \tag{1}$$

for $n\ge n_0(\epsilon)$ and all y. Thus $F_n(y)\to F(y)$ at all points of continuity y of $F(y)$. Now assume $F_n\to F$ in distribution. If $\rho(F_n,F)$ does not converge to zero, there exists a subsequence (while we also call $\{\,F_n\,\}$) and a value $\epsilon>0$ such that $\rho(F_n,F)\ge 2\epsilon>0$, and thus real values y_n such that

$$F(y_n-\epsilon)-\epsilon \;\ge\; F_n(y_n) \quad\text{or}\quad F_n(y_n) \;\ge\; F(y_n+\epsilon)+\epsilon \tag{2}$$

with, choosing a further subsequence if necessary, one of the two inequalities in (2) holding for all n. Since F_n,F are distribution functions, there cannot exist a subsequence $y_{n_k}\to\infty$ or $y_{n_k}\to-\infty$, so that y_n are bounded. Hence there exists a further subsequence (which we also call F_n,y_n) such that $y_n\to y$ for $y\in R$. Recall that if $F_n\to F$ and $y_n\to y$, then

$$F(y-) \;\le\; \liminf_{n\to\infty} F_n(y_n) \;\le\; \limsup_{n\to\infty} F_n(y_n) \;=\; F(y)$$

Then by (2)

$$F(y-\epsilon)-\epsilon \;\ge\; F(y-) \quad\text{or}\quad F(y) \;\ge\; F(\,(y+\epsilon)-\,)+\epsilon$$

for some $\epsilon>0$, either of which provides a contradiction.

5. (a) $E(|X_k|^{2-\delta})=2\int_1^\infty y^{2-\delta}y^{-3}dy<\infty$ for $\delta>0$ but not for $\delta\le 0$, etc.

(b) Set $S_n=(X_1+X_2+\ldots+X_n)/a_n$ for $a_n=c\sqrt{n\log n}$ for some constant $c>0$. Then $E\big(e^{i\theta S_n}\big)=\phi(\theta/a_n)^n$ for $\phi(\theta)=E\big(e^{i\theta X_1}\big)$. Since the X_k are symmetrically distributed, $\phi(\theta)=\phi(-\theta)$ and, for $\theta>0$,

$$\begin{aligned} 1-\phi(\theta) \;&=\; \int_{|y|\ge 1}\big(1-e^{i\theta y}\big)\,\frac{dy}{|y|^3} \;=\; 2\int_1^\infty \big(1-\cos(\theta y)\big)\frac{dy}{y^3} \\ &=\; 2\theta^2\int_\theta^\infty (1-\cos y)\frac{dy}{y^3} \end{aligned}$$

By L'Hopital's rule (or otherwise), $\lim_{\theta\to 0}\int_\theta^\infty (1-\cos y)y^{-3}dy \;/\; \log(1/\theta) = 1/2$, so that

$$\lim_{\theta\to 0}\frac{1-\phi(\theta)}{\theta^2\log(1/\theta)} \;=\; 1$$

In particular, for fixed $\theta>0$,

$$\begin{aligned} 1-\phi(\theta/a_n) \;&\sim\; \frac{\theta^2}{a_n^2}\log\left(\frac{a_n}{\theta}\right) \;=\; \frac{\theta^2}{c^2 n\log n}\log\left(\frac{c\sqrt{n\log n}}{\theta}\right) \\ &=\; \frac{\theta^2}{2c^2 n\log n}\big(\log n+\log\log n+2\log(c/\theta)\big) \;\sim\; \frac{\theta^2}{2c^2 n} \end{aligned}$$

as $n\to\infty$. Thus

$$\log\phi(\theta/a_n) \;=\; \log\big(1-(1-\phi(\theta/a_n))\big) \;=\; -\big(1-\phi(\theta/a_n)\big)+O(1/n^2)$$

and

$$\log\big(E\big(e^{i\theta S_n}\big)\big) \;=\; n\log\phi(\theta/a_n) \;=\; -n\big(1-\phi(\theta/a_n)\big)+O(1/n) \;\to\; -\frac{\theta^2}{2c^2}$$

Thus $E\left(e^{i\theta S_n}\right)\to\exp(-\theta^2/2)$ for all θ for $c=1$, which implies that S_n converges in distribution to a standard normal distribution.

6. (a) Here $S_n=(X_1+X_2+\ldots+X_n)/\sqrt{n}$ where

$$\phi_k(\theta)=E\left(e^{i\theta X_k}\right) \;=\; \frac{e^{i\theta\sqrt{k}}}{2k}+\left(1-\frac{1}{k}\right)+\frac{e^{-i\theta\sqrt{k}}}{2k} \;=\; 1 \;-\; \frac{1-\cos(\theta\sqrt{k})}{k} \qquad (1)$$

Then $E\left(e^{i\theta S_n}\right)=\prod_{k=1}^n\phi_k\big(\theta/\sqrt{n}\big)$. We give two proofs of part (a), the first using Theorem 28.3 in the text, which was proven in class, and the second arguing directly from $\Phi_n(\theta)=E\left(e^{i\theta S_n}\right)$.

Proof I. Thus $S_n=\sum_{k=1}^n X_{nk}$ for $X_{nk}=X_k/\sqrt{n}$. Since $E(X_{nk})=0$, $E(X_{nk}^2)=E(X_k^2)/n=1/n$, and $E(S_n^2)=1$, the S_n are the row sums of a triangular array in the sense of page 272 of the text. Thus by Theorem 28.3

$$\lim_{n\to\infty}\left(E\left(e^{i\theta S_n}\right)-\exp\left(\sum_{k=1}^n\left(\phi_k\left(\frac{\theta}{\sqrt{n}}\right)-1\right)\right)\right)=0 \qquad (2)$$

for all θ. By (1)

$$\sum_{k=1}^n\left(\phi_k\left(\frac{\theta}{\sqrt{n}}\right)-1\right) \;=\; -\sum_{k=1}^n\frac{1-\cos\big(\theta\sqrt{k/n}\big)}{k} \;=\; -\frac{1}{n}\sum_{k=1}^n\left(\frac{1-\cos\big(\theta\sqrt{k/n}\big)}{k/n}\right)$$

Since $1 - \cos(y) \le (1/2)y^2$ for all y, the term within the last set of parentheses in the display is uniformly bounded for fixed θ and $1 \le k \le n$. Hence for all θ

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left(\phi_k \left(\frac{\theta}{\sqrt{n}} \right) - 1 \right) \;=\; -\int_0^1 \frac{1 - \cos\big(\theta\sqrt{y}\big)}{y}\, dy \;=\; -2\int_0^1 \frac{1 - \cos\big(\theta y\big)}{y}\, dy$$

Hence by (2)

$$\lim_{n\to\infty} E\left(e^{i\theta S_n}\right) \;=\; \psi(\theta) \;=\; \exp\left(-2\int_0^1 \frac{1-\cos\big(\theta y\big)}{y}\, dy \right) \tag{3}$$

$$= \exp\left(-2\int_0^\theta \frac{1-\cos(y)}{y}\, dy \right)$$

Proof II. Since $1 - \cos(y) \le (1/2)y^2$ for all y, it follows from (1) that

$$1 - \phi_k\left(\frac{\theta}{\sqrt{n}}\right) \;=\; \frac{1}{n}\left(\frac{1 - \cos\big(\theta\sqrt{k/n}\big)}{k/n} \right) \;=\; O(1/n)$$

uniformly for $1 \le k \le n$ and

$$\log \phi_k\left(\frac{\theta}{\sqrt{n}}\right) \;=\; \log\left(1 - \left(1 - \phi_k\left(\frac{\theta}{\sqrt{n}}\right)\right)\right) \;=\; -\left(1 - \phi_k\left(\frac{\theta}{\sqrt{n}}\right)\right) + O\left(\frac{1}{n^2}\right)$$

where the error term is uniform for $1 \le k \le n$. Hence the logarithms below exist and

$$\log\left(E\left(e^{i\theta S_n}\right)\right) \;=\; \sum_{k=1}^{n} \log\phi_k\left(\frac{\theta}{\sqrt{n}}\right) \;=\; -\sum_{k=1}^{n}\left(1 - \phi_k\left(\frac{\theta}{\sqrt{n}}\right) + O\left(\frac{1}{n^2}\right)\right)$$

$$= \left(-\frac{1}{n}\sum_{k=1}^{n}\left(\frac{1-\cos(\theta\sqrt{k/n})}{k/n} \right)\right) + O\left(\frac{1}{n}\right)$$

$$\to\; -\int_0^1 \frac{1-\cos(\theta\sqrt{y})}{y}\, dy \;=\; -2\int_0^1 \frac{1-\cos(\theta y)}{y}\, dy$$

and (3) follows as above.

(b) The function $\psi(\theta)$ is continuous in θ and $\lim_{\theta\to 0}\psi(\theta) = 1$. Thus, by the Lévy continuity theorem, there exists a distribution function $F(y)$ such that $F_n(y) = P(S_n \le y) \to F(y)$ at all continuity points of $F(y)$ and $\psi(\theta) = \int e^{i\theta y} F(dy)$.

If $F(y)$ were normal, then $\psi(\theta) = \psi_N(\theta) = \exp(i\mu\theta - (1/2)\sigma^2\theta^2)$ for some choice of real constants μ and σ. However

$$\frac{d}{d\theta}\log\psi(\theta) = -2\frac{1-\cos\theta}{\theta} \qquad \text{and} \qquad \frac{d}{d\theta}\log\psi_N(\theta) = i\mu - \sigma^2\theta$$

Note that $(d/d\theta)\log\psi(\theta) = 0$ if and only if $\theta = 2n\pi$ for some $n \ne 0$, which is not true for $(d/d\theta)\log\psi_N(\theta)$ for any values of μ and σ. Thus $F(y)$ cannot be a normal distribution.