Ma 551 — Advanced Probability

Take-Home Final — Due December 16, 2009

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A fixed probability space $(\Omega, \mathcal{F}, P)$ is assumed in many of the problems. Here r.v. means random variable, i.r.v. means independent r.v.s, and i.i.d. means independent and identically distributed (random variables). Problems in text are from Patrick Billingsley, *Probability and Measure*, 3rd edn, John Wiley & Sons, 1995. Feel free to use results from one problem as theorems for other problems.

Eight problems on three pages.

1. Suppose $X_n = \prod_{j=1}^n Y_j$ where $\{ Y_j \}$ are i.r.v.s with $Y_j > 0$ a.s. and $E(Y_j) = 1$.

(a) Find σ-algebras $\mathcal{F}_n \uparrow$ such that $\{ (X_n, \mathcal{F}_n) \}$ is a martingale. Show that $\lim_{n\to\infty} X_n = X$ a.s. where $X \geq 0$ a.s. and $E(X) \leq 1$.

(b) Assume Y_j are i.i.d. with $Y_j \approx U(0,2)$. (That is, Y_j is uniformly distributed in $(0,2)$.) Thus $E(Y_j) = 1$ and $Y_j > 0$ a.s. Prove that $X_n \to 0$ a.s.

(*Hint*: Consider taking logarithms.)

2. Let $\{ (X_n, \mathcal{F}_n) \}$ be a submartingale with $E(X_n) = E(X_1)$ for all $n \geq 1$. Prove that $\{ (X_n, \mathcal{F}_n) \}$ is a martingale.

3. Let $\mathcal{B}_n = \mathcal{B}(X_1, \ldots, X_n)$ for r.v.s $X_1, \ldots . X_n$ be the smallest σ-algebra with respect to which $X_1, \ldots, X_n$ are measurable. Show that any $\mathcal{B}_n$-measurable r.v. Y can be written

$$Y(\omega) = \phi\big(X_1(\omega), \ldots, X_n(\omega)\big) \tag{1}$$

for some Borel function $\phi(y)$ on R^n.

(*Hints*: Prove this first for $Y = I_E$ for events $E \in \mathcal{B}_n$ and then approximate Y by simple r.v.s $Y_1 = \sum_{k=1}^m c_k I_{E_k}$. To prove (1) for $Y = I_E$, note that $H = (X_1, \ldots, X_n) : \Omega \to R^n$ is measurable in the sense that, given λ_i for $1 \leq i \leq n$, $\{ \omega : X_i(\omega) \leq \lambda_i \text{ for } 1 \leq i \leq n \} \in \mathcal{B}_n$. If $\mathcal{C}$ is the class of all sets $A \in \mathcal{B}(R^n)$ such that $H^{-1}(A) = \{ \omega : H(\omega) \in A \} \in \mathcal{B}_n$, show that $\mathcal{C} = \mathcal{B}(R^n)$ and $H^{-1}(\mathcal{C}) = \mathcal{B}_n$. Put all of this together to conclude (1).)

Remark. Let $Z, X_1, \ldots, X_n$ be r.v.s with $E(|Z|) < \infty$. Since $E(Z \mid \mathcal{B}_n)$ is $\mathcal{B}_n$-measurable,

$$E\big(Z \mid \mathcal{B}(X_1, \ldots, X_n)\big) = \phi(X_1, \ldots, X_n) \ a.s. \tag{2}$$

for some Borel function $\phi(y)$ on R^n.

4. Let $\mathcal{F}_n \uparrow \mathcal{F}$ be σ-algebras $\mathcal{F}_n \subseteq \mathcal{F}$ for the probability space $(\Omega, \mathcal{F}, P)$. Let $Z_n = E(f \mid \mathcal{F}_n)$ for $f \in L^1(\Omega, \mathcal{F}, P)$. Prove that $Z_n \to f$ a.s.

(*Hint*: By definition, $\Gamma =$ (set union) $\cup\, \mathcal{F}_n$ is a generating semi-ring for $\mathcal{F}$. Prove first for $f = I_E$ and approximate by simple r.v.s.)

Remarks. In Bayesian statistics, $F(y) = P(f \le y)$ can be viewed as the "prior distribution" of an unknown parameter $f(\omega)$. If $\mathcal{F}_n = \mathcal{B}(X_1, \ldots, X_n)$ for r.v.s X_k, then, by the preceding problem, $Z_n = E(f \mid \mathcal{B}(X_1, \ldots, X_n)) = \phi_n(X_1, \ldots, X_n)$ for some Borel function $\phi_n(y)$ on R^n.

The conditional expectations Z_n are the orthogonal projections of f onto $L^2(\mathcal{F}_n)$, or equivalently onto the linear space of all L^2 r.v.s of the form $\phi(X_1, \ldots, X_n)$. This projection called the *Bayes estimator* of f (for a quadratic loss function) given the finite sample $X_1, \ldots, X_n$. In this context, $Z_n \to f$ a.s. is called the *asymptotic consistency* of the Bayes estimator of f for the sample $\{ X_1, X_2, \ldots \}$, and shows that the true value $f(\omega)$ of f can be retrieved from $\{ X_1, X_2, \ldots \}$ with probability one.

5. Let X, Y be two r.v. such that

$$P(Y \le \lambda,\, X \le \mu) \;=\; F_{Y,X}(\lambda, \mu) \;=\; \int_{-\infty}^{\lambda} \int_{-\infty}^{\mu} f(z, w)\, dz dw$$

for all λ, μ, where $f(x, y) \ge 0$ and $\iint_{R^2} f(z, w) dz dw = 1$. Assume $E(|Y|) < \infty$ (I forgot to include this condition earlier). Find the Borel function $\phi(x)$ on R corresponding to $E(Y \mid \mathcal{B}(X)) = \phi(X)$ in (2) in terms of $f(x, y)$. Verify your result.

6. Let X_i be i.i.d. with $\phi(\theta) = E\left(e^{\theta X_i}\right) < \infty$ for all real θ, $E(X_i) < 0$, and $P(X_i > 0) > 0$. Let $S_n = X_1 + X_2 + \ldots + X_n$.

(a) Set $Z_n(\theta) = \exp(\theta S_n)\phi(\theta)^{-n}$ for $\theta \in R$. Show that $\{ (Z_n(\theta), \mathcal{F}_n) \}$ is a martingale for some set of σ-algebras $\mathcal{F}_n \uparrow$ (and say what your $\mathcal{F}_n$ are).

(b) Prove that there exists a value $\theta_0 > 0$ such that

$$\Pr\left(\max_{1 \le n < \infty} S_n \ge \lambda \right) \;\le\; e^{-\theta_0 \lambda} \tag{3}$$

for all $\lambda > 0$. (*Hint*: Consider $\phi'(0)$ and $\lim_{\theta \to \infty} \phi(\theta)$.)

Remark. Since $E(X_i) < 0$, $\lim_{n \to \infty} S_n = -\infty$ a.s. by the strong law of large numbers. Equation (3) gives an upper bound on the largest positive value that S_n can attain before converging to $-\infty$, and can be used to estimate your probability of going broke before massively winning in a favorable gambling game.

7. Suppose that $p_{ij} \geq 0$ for integers $-\infty < i, j < \infty$ with $\sum_k p_{ik} = 1$ and $p_{ii} < 1$ for all i. Let $\{ X_n \}$ be integer-valued r.v.s such that $X_0 = 0$ and

$$P(X_{n+1} = j \mid \mathcal{B}(X_1, X_2, \ldots, X_n)) \;=\; P(X_{n+1} = j \mid X_n = i) \;=\; p_{ij} \; a.s.$$

for $i = X_n(\omega)$, $n \geq 0$, and all $j \in J$ where J is the set of integers. (The stochastic process X_n is called the *Markov chain* generated by p_{ij} with $X_0 = 0$. That such r.v.s exist follows easily from the Kolmogorov Consistency Theorem.)

A function $\phi(i)$ on J is called *p-harmonic* if $\phi(i) = \sum_{k=-\infty}^{\infty} p(i,k)\phi(k)$ for all $i \in J$. Suppose that there exists a p-harmonic function $\phi(i) \geq 0$ on J that is strictly increasing and unbounded on J. (That is, $\phi(i) < \phi(i+1)$ for all i and $\sup_i \phi(i) = \infty$.) Prove that

$$\lim_{n\to\infty} X_n(\omega) = -\infty \qquad \text{a.s.}$$

This shows that there is a connection between the harmonic functions of an infinite matrix p_{ij} and the sample behavior of X_n. (*Hint*: Is $\phi(X_n)$ a martingale or submartingale for appropriate σ-algebras $\mathcal{F}_n$?)

Remarks. Suppose $p_{i,i-1} = p_i$, $p_{ii} = q_i$, and $p_{i,i+1} = r_i$ where $p_i, q_i, r_i > 0$ and $p_i + q_i + r_i = 1$. The resulting Markov chain is called variously a *nearest-neighbor random walk* or a *birth and death process*.

In this case, $\phi(i) = p_i\phi(i-1) + q_i\phi(i) + r_i\phi(i+1)$ implies $\phi(i+1) - \phi(i) = (p_i/r_i)\big(\phi(i) - \phi(i-1)\big)$. If $\phi(1) - \phi(0) > 0$, then $\phi(i)$ is strictly increasing. If X_n has a bias to go to the left (that is, $p_i > r_i$), then $\phi(i)$ is convex. This suggests (but does not prove — that is up to you) a possible connection between $\phi(i) \geq 0$ and $\phi(i) \uparrow \infty$ and a strong tendency for X_n to go to the left.

8. Let $\{ (X_n, \mathcal{F}_n) \}$ be a nonnegative submartingale. Prove that we can write

$$X_n \;=\; Y_n \;+\; Z_n \tag{4}$$

where $\{ (Y_n, \mathcal{F}_n) \}$ is a martingale, $0 \leq Z_n \uparrow$ a.s., and Z_n is *predictable* — that is, $\mathcal{F}_{n-1}$-measurable. Show that the decomposition (4) is a.s. unique given $Y_0 = 0$ a.s. for martingales Y_n and predictable processes Z_n.

(*Hint*: Consider $\Delta_{X,n} = X_n - X_{n-1}$.)

Remarks. Since $E(\Delta_{X,n} \mid \mathcal{F}_{n-1}) = \Delta_{Z,n} = Z_n - Z_{n-1}$, the latter r.v. is the minimum-variance unbiased predictable estimator of $\mathcal{F}_n$-measurable $X_n - X_{n-1}$, with Y_n viewed as a sum of errors. Equation (4) is a discrete version of what is called the *Doob-Meyer decomposition*. Counting-process theory in survival analysis, as well as some forms of optimal control theory in Engineering, are based on this decomposition.