Special Topics in Comp Bio

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The gamma distribution:

This is a distribution for $x \geq 0$ with density

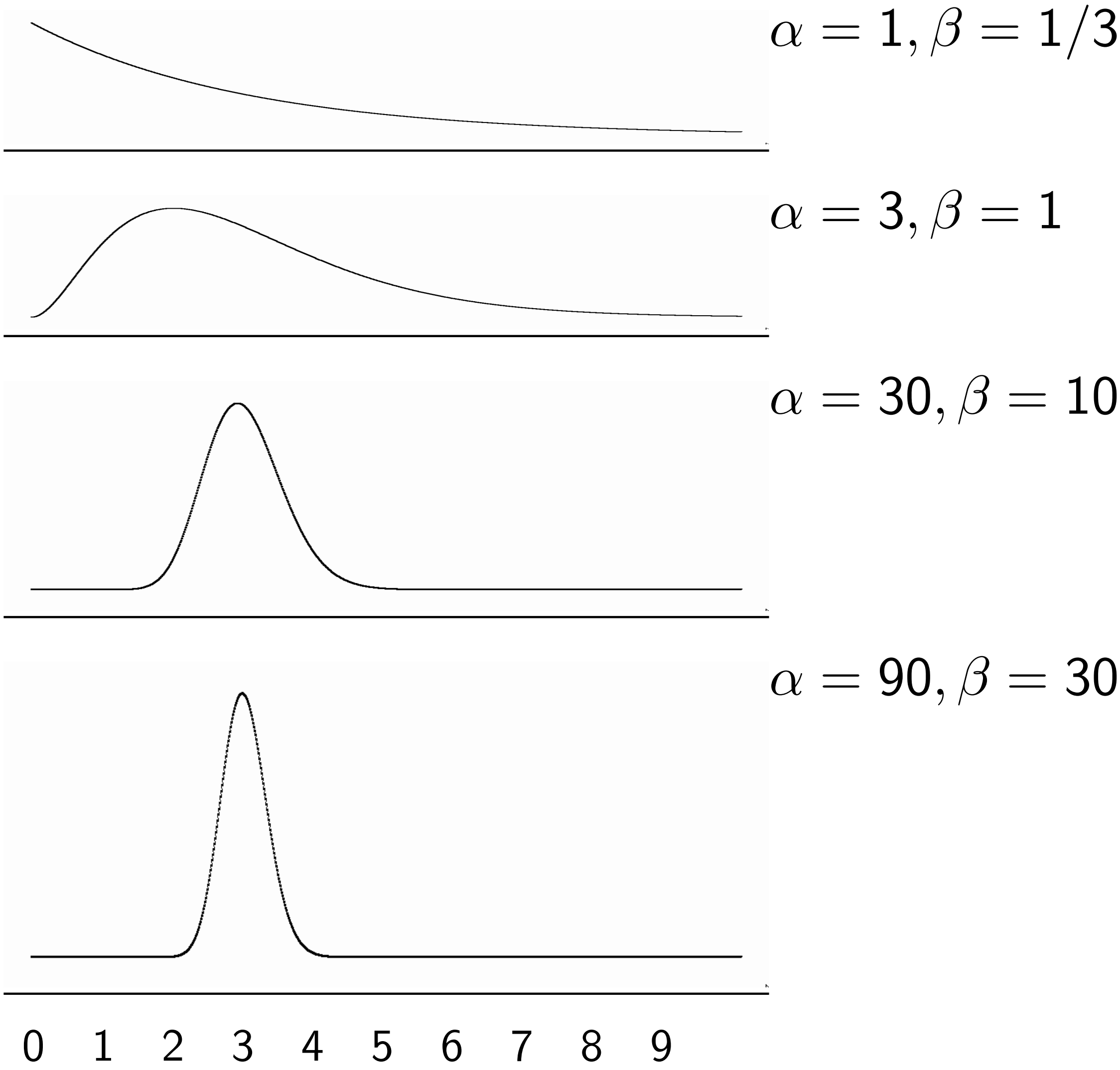
$$\frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \exp(-\beta x) \qquad 0 \leq x < \infty$$

Here $\alpha, \beta > 0$ and $\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} \exp(-\beta y)\, dy$.

We say $X \approx \mathrm{Gam}(\alpha, \beta)$ if a random variable X has this density. For $\alpha = 1$, $\mathrm{Gam}(1, \beta)$ is the exponential distribution with rate β:

$$\beta \exp(-\beta x) \qquad 0 \leq x < \infty$$

Some example densities:



If $X \approx \text{Gam}(\alpha, \beta)$,

$$E(X) = \alpha/\beta, \qquad \text{Var}(X) = \alpha/\beta^2$$

In general

$$\text{Gam}(\alpha, \beta) \approx (1/\beta)\,\text{Gam}(\alpha, 1)$$

(that is, β is a rate parameter).
Gamma distributions can be scaled by setting

$$X_v = \text{Gam}\left(\frac{1}{v}, \frac{1}{v}\right), \qquad Y = \theta X_v$$

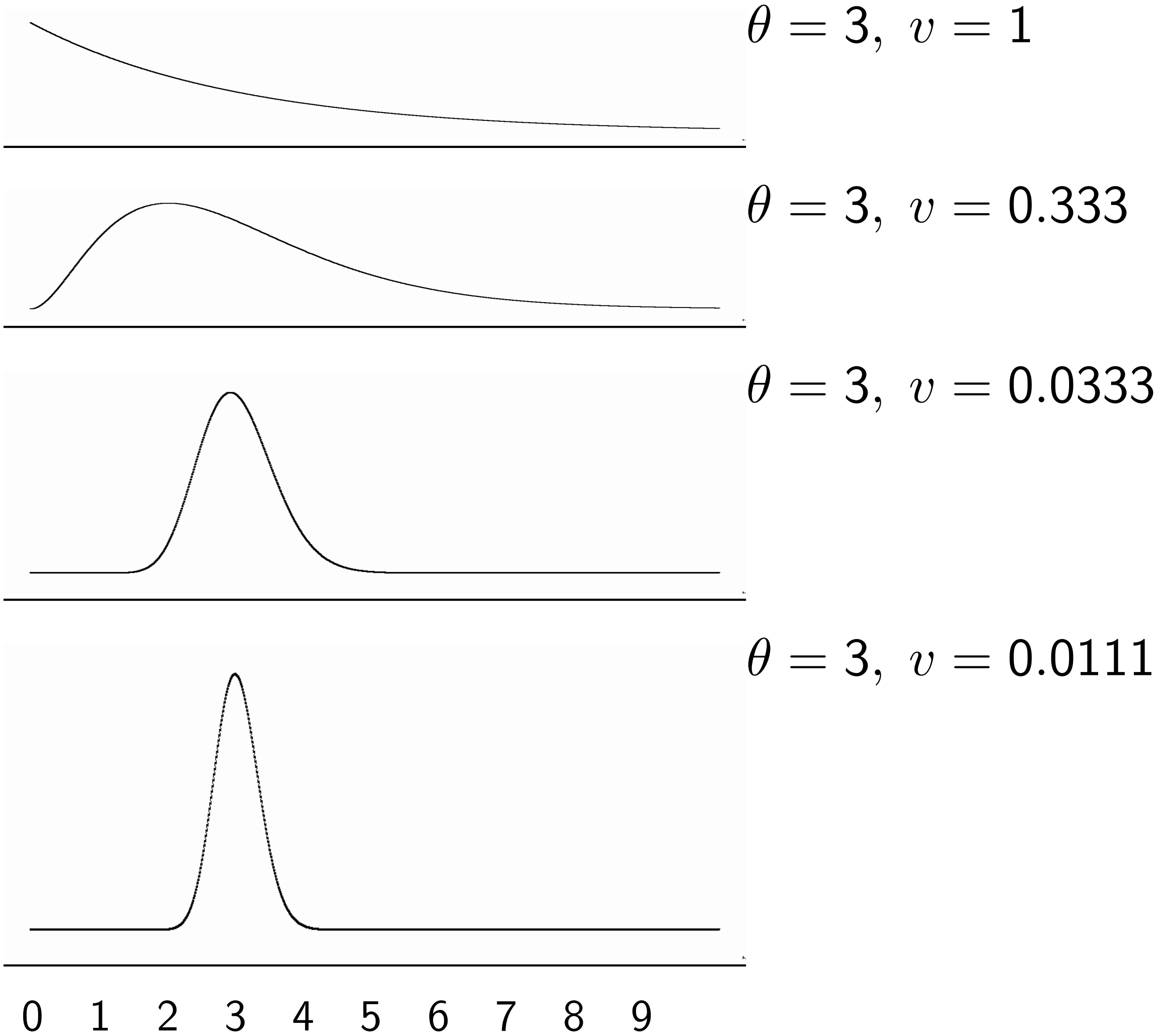
Then

$$E(X_v) = 1, \qquad \text{Var}(X_v) = v,$$

$$E(Y) = \theta, \qquad \text{Var}(Y) = \theta^2 v$$

which allows the modeling of arbitrary random $X > 0$ in terms of $E(X)$ and $\text{Var}(X)$:

The same densities in θ and v coordinates:



Some other important properties of gamma variables:

(i) If $X_1 \approx \text{Gam}(\alpha_1, \beta)$ and $X_2 \approx \text{Gam}(\alpha_2, \beta)$ and X_1 and X_2 are independent, then

$$X_1 + X_2 \approx \text{Gam}(\alpha_1 + \alpha_2, \beta)$$

That means that $T_k = \text{Gam}(k, \beta)$ can be viewed as the waiting time for k independent events, where the k events must occur in sequence and each has an exponential waiting time $\text{Gam}(1, \beta)$.
The resulting distribution

$$\text{Gam}(k, \beta) = \frac{\beta^k}{(k-1)!} x^{k-1} \exp(-\beta x)$$

is called the *Erlang* distribution in queueing theory.

(ii) If $z \approx N(0,1)$, then z^2 is Gam(1/2, 1/2). Thus

$$\chi^2_n \;\approx\; z_1^2 + z_2^2 + \ldots + z_n^2 \;\approx\; \text{Gam}(n/2, 1/2)$$

This means that chi-square distributions in statistics are special cases of gamma distributions.

(iii) An interesting use of gamma distributions is Fisher's method of combining the results of different experiments. (Nowadays this would be called "meta-analysis".)

Suppose that you conducted four different experiments and concluded that none were significant, with P-values

$$P_1 = 0.07,\; P_2 = 0.18,\; P_3 = 0.09,\; P_4 = 0.14$$

Taken together, are these enough to conclude significance, assuming that you are not able to combine all the data and analyze them together?

Fishers idea is as follows. The first step is to combine the four P-values into a single score, for which one can assign a single P-value. A natural choice is

$$T = P_1 P_2 P_3 P_4$$

for which $T_{\text{obs}} = (0.07)(0.18)(0.09)(0.14) = 0.0001430$ Is this significantly small, given that it is the product of P-values for 4 experiments? The key idea is that, if a null hypothesis is true, then the P-value itself is uniformly distributed in $(0, 1)$. Thus

$$P(-\log(P_i) \geq t) = P(P_i \leq e^{-t}) = e^{-t}$$

This means that each $-\log(P_i) \approx \text{Gam}(1, 1)$ given H_0. Thus

$$-\log(T) = -\sum_{i=1}^{4} \log(P_i) \;\approx\; \text{Gam}(4, 1)$$

In Fisher's day, there were χ^2 tables but no computers or statistical calculators. However

$$\text{Gam}(4,1) \approx (1/2)\,\text{Gam}(8/2,1/2) \approx (1/2)\chi^2_8$$

Hence the overall P-value is

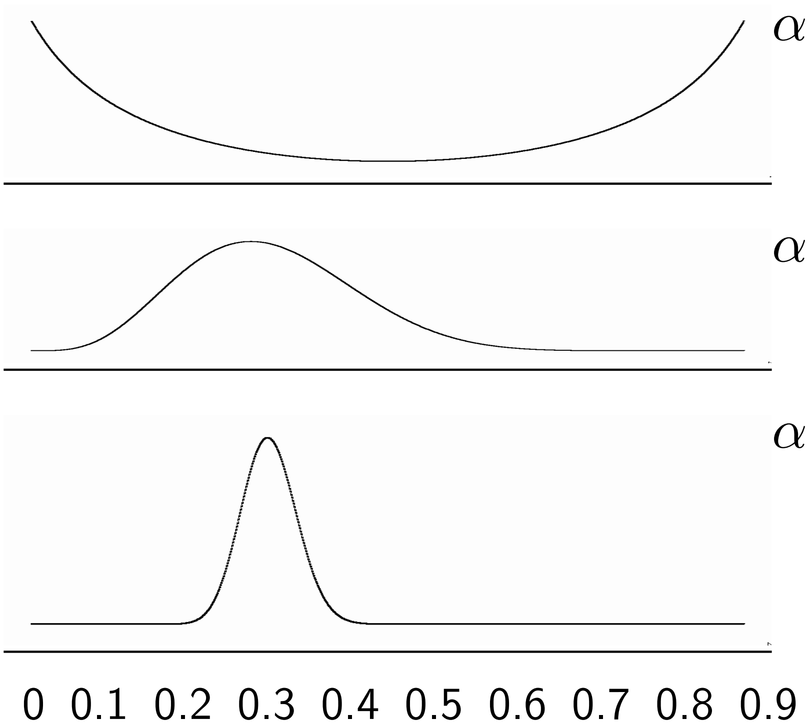
$$P = \Pr(\chi^2_8 \geq -2\log(T)) = \Pr(\chi^2_8 \geq 17.71) = 0.024$$

Thus the combined effect of the four experiments is significant.

The beta distribution: This is a distribution with density

$$C\,x^{\alpha-1}(1-x)^{\beta-1}, \qquad 0 \le x \le 1$$

where $C = \Gamma(\alpha+\beta)/\big(\Gamma(\alpha)\Gamma(\beta)\big)$. Some examples are:



$\alpha = \beta = 1/2$

$\alpha = 5, \beta = 10$

$\alpha = 50, \beta = 100$

We say $X \approx \text{Beta}(\alpha, \beta)$ if X has this density. Then $\text{Beta}(1, 1)$ is uniform and

$$E(X) = \frac{\alpha}{\alpha + \beta}, \quad \text{Var}(X) = \frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)}$$

If $\theta = \alpha/(\alpha + \beta)$ and $V = \alpha + \beta + 1$, then

$$E(X) = \theta, \qquad \text{Var}(X) = \frac{\theta(1 - \theta)}{V}$$

One can show that if $Z \approx \text{Beta}(\alpha, \beta)$, then

$$Z \approx \frac{X_1}{X_1 + X_2}$$

where $X_1 \approx \text{Gam}(\alpha, r)$, $X_2 \approx \text{Gam}(\beta, r)$, and X_1 and X_2 are independent. This implies

$$\frac{Z}{1 - Z} \approx \frac{X_1}{X_2} \approx \frac{\text{Gam}(\alpha, r)}{\text{Gam}(\beta, r)} \approx \frac{\chi^2(2\alpha)}{\chi^2(2\beta)}$$

Thus if $Z \approx \text{Beta}(\alpha, \beta)$

$$\frac{Z}{1-Z} \;\approx\; \frac{\alpha}{\beta}\,\frac{\chi^2(2\alpha)/2\alpha}{\chi^2(2\beta)/2\beta} \;\approx\; \frac{\alpha}{\beta}F(2\alpha, 2\beta)$$

so that $Z \approx \text{Beta}(\alpha, \beta)$ can be written in terms of an F-distribution and vice versa. This is in fact how F-distribution P-values are calculated in many statistical packages, since the F-distribution density itself has polynomial decay at infinity.

The beta density can also be written

$$f(x) = Cx_1^{\alpha-1}x_2^{\beta-1}$$

where (x_1, x_2) are on the line $x_1 + x_2 = 1$ for $x_1 \geq 0, x_2 \geq 0$. This is an equivalent way of looking at a beta density, as long as you are careful when you are integrating: The "dx" on the line $x_1 + x_2 = 1$ is $1/\sqrt{2}$ of the size of "dx" for x on the real line.

The Dirichlet distribution: Once one gets used to this, one can generalize the beta density to more that two variables: For example, with a three-dimensional density

$$f(x) = Cx_1^{\alpha-1}x_2^{\beta-1}x_3^{\gamma-1}x_4^{\delta-1}$$

on the simplex $x_1 + x_2 + x_3 + x_4 = 1$, $x_i \geq 0$, for parameters $\alpha, \beta, \gamma, \delta > 0$.

This is called a *Dirichlet density* and has very similar properties to a beta density. In fact, if random variables X_1, X_2, X_3, X_4 have the above Dirichlet distribution (so that $X_1 + X_2 + X_3 + X_4 = 1$), then the X_i can be represented

$$X_i \approx \frac{Y_i}{Y_1 + Y_2 + Y_3 + Y_4} \qquad (1 \le i \le 4)$$

where the $Y_i \approx \text{Gam}(\alpha_i, r)$ are independent gamma-distributed random variables where $(\alpha_1, \dots, \alpha_4) = (\alpha, \beta, \gamma, \delta)$.