

### Third Midterm

**General Instructions:** Read the statement of each problem carefully. Do only what is requested—nothing more and nothing less. Of course you need not show any work for the multiple choice or the TRUE/FALSE questions. For the questions that require a written answer, provide a *complete solution*. If you only write the answer then you will not get full credit.

Be sure to ask questions if anything is unclear. This exam is worth 100 points.

- (10 points) 1. Calculate the tangent plane to the graph of the function  $f(x, y) = 2x^2y + x^2$  at the point  $(1, 2, 5)$ .

$$\underline{n} = \langle f_x, f_y, -1 \rangle = \langle 4xy + 2x, 2x^2, -1 \rangle$$

$$\text{At } (1, 2, 5), \underline{n} = \langle 10, 2, -1 \rangle$$

Plane is

$$\langle 10, 2, -1 \rangle \cdot \langle x-1, y-2, z-5 \rangle = 0$$

$$10x - 10 + 2y - 4 - z + 5 = 0$$

$$10x + 2y - z = 9.$$

- (10 points) 2. Locate and identify the local maxima and local minima and saddle points of the function  $f(x, y) = x^3 + xy + 3y^2 - 15x + 6$ .

$$\nabla f = \langle 0, 0 \rangle \Rightarrow \langle 3x^2 + y - 15, x + 6y \rangle = \langle 0, 0 \rangle$$

$$3x^2 + y = 15$$

$$x + 6y = 0 \Rightarrow x = -6y$$

$$\text{So } 3(-6y)^2 + y = 15$$

$$108y^2 + y = 15 \Rightarrow 108y^2 + y - 15 = 0$$

$$y = \frac{-1 \pm \sqrt{1 + 6480}}{216} = \frac{-1 \pm \sqrt{6481}}{216}$$

$$x = \frac{6 \mp 6\sqrt{6481}}{216}, \text{ So } \left( \frac{6 - 6\sqrt{6481}}{216}, \frac{-1 + \sqrt{6481}}{216} \right) = A$$

$$\left( \frac{6 + 6\sqrt{6481}}{216}, \frac{-1 - \sqrt{6481}}{216} \right) = B$$

$$D = 36x - 1$$

A is a saddle

B is a min

- (10 points) 3. Find the extrema of the function  $f(x, y) = y - x^2$  subject to the constraint  $g(x, y) = x^2 + y^2 = 4$ .

$$\nabla f = \lambda \nabla g \quad \langle -2x, 1 \rangle = \lambda \langle 2x, 2y \rangle$$

$$-2x = 2\lambda x, \quad 1 = 2\lambda y$$

$$x^2 + y^2 = 4$$

$$-2x(1 + \lambda) = 0 \Rightarrow x = 0 \quad \text{or} \quad \lambda = -1$$

If  $x = 0$ , then  $y = \pm 2$ . Crit. pts. are  $(0, 2), (0, -2)$ .

If  $\lambda = -1$ , then  $y = -\frac{1}{2}$  so  $x^2 + \frac{1}{4} = 4$

$$x^2 = \frac{15}{4} \Rightarrow x = \pm \frac{\sqrt{15}}{2}$$

Crit. pts. are  $(\frac{\sqrt{15}}{2}, -\frac{1}{2}), (-\frac{\sqrt{15}}{2}, -\frac{1}{2})$

$$f(0, 2) = 2, \quad f(0, -2) = -2, \quad f(\frac{\sqrt{15}}{2}, -\frac{1}{2}) = -\frac{1}{2} - \frac{15}{4} = -\frac{17}{4} \text{ min}$$

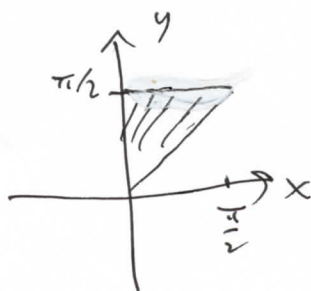
max

$$f(-\frac{\sqrt{15}}{2}, -\frac{1}{2}) = -\frac{1}{2} - \frac{15}{4} = -\frac{17}{4} \text{ min}$$

(10 points) 4. Calculate the integral

$$\begin{aligned}
 & \int_1^3 \int_2^5 y^2 x - x^2 y \, dy \, dx \\
 &= \int_1^3 \left[ \frac{y^3 x}{3} - \frac{x^2 y^2}{2} \right]_{y=2}^{y=5} dx \\
 &= \int_1^3 \left( \frac{125}{3} x - \frac{25}{2} x^2 \right) - \left( \frac{8}{3} x - 2x^2 \right) dx \\
 &= \int_1^3 \frac{117}{3} x - \frac{21}{2} x^2 dx \\
 &= \left[ \frac{117}{6} x^2 - \frac{7}{2} x^3 \right]_1^3 = \left( \frac{117 \cdot 9}{6} - \frac{7 \cdot 27}{2} \right) - \left( \frac{117}{6} - \frac{7}{2} \right) \\
 &= 128.
 \end{aligned}$$

(10 points) 5. Reverse the order of integration in order to evaluate the following double integral:



$$\begin{aligned}
 & \int_0^{\pi/2} \int_x^{\pi/2} \frac{\cos 2y}{y} \, dy \, dx \\
 &= \int_0^{\pi/2} \int_0^y \frac{\cos 2y}{y} \, dx \, dy \\
 &= \int_0^{\pi/2} \frac{\cos 2y}{y} x \Big|_{x=0}^{x=y} dy \\
 &= \int_0^{\pi/2} \cos 2y \, dy = \left[ -\frac{\sin 2y}{2} \right]_0^{\pi/2} \\
 &= -0 + 0 = 0.
 \end{aligned}$$

- (10 points) 6. Calculate the volume of the solid that lies below the graph of  $f(x, y) = x^2 + 2y^4$  and over the rectangle  $[0, 2] \times [0, 1]$  in the  $x$ - $y$  plane.

$$\begin{aligned} \int_0^2 \int_0^1 x^2 + 2y^4 \, dy \, dx &= \int_0^2 \left[ x^2 y + \frac{2}{5} y^5 \right]_{y=0}^{y=1} dx \\ &= \int_0^2 x^2 + \frac{2}{5} \, dx = \left[ \frac{x^3}{3} + \frac{2}{5} x \right]_0^2 = \frac{8}{3} + \frac{4}{5} = \frac{52}{15}. \end{aligned}$$

- (10 points) 7. Describe the symmetries (in the  $x$ -axis, the  $y$ -axis, and the origin) of the curve  $r = \sin 2\theta$ .

$x$  axis symmetry:  $\theta \mapsto -\theta$   
 $r = \sin(-\theta) = -\sin \theta$  X

$y$  axis symmetry:  $\theta \mapsto \pi - \theta$   
 $r = \sin(2(\pi - \theta)) = \sin(2\pi - 2\theta)$   
 $= \sin(-2\theta) = -\sin 2\theta$  X

origin:  $r \mapsto -r$   
 $-r = \sin 2\theta$  X

$\theta \mapsto \pi + \theta$   
 $r = \sin(2(\pi + \theta)) = \sin(2\pi + 2\theta) = \sin 2\theta$  ✓

So symmetry is the origin only.

I note that this graph does have symmetry in the  $x$ - and  $y$ -axes, but the tests don't detect them.

- (10 points) 8. What are the parametric equations of the normal line to the surface  $y^2 - 2x^2 + 4z^2 = 3$  at the point  $(1, 1, 1)$ ?

$$\underline{n} = \langle -4x, 2y, 8z \rangle$$

$$\text{At } (1, 1, 1), \underline{n} = \langle -4, 2, 8 \rangle.$$

Line is :

$$x = 1 - 4t$$

$$y = 1 + 2t$$

$$z = 1 + 8t$$

- (10 points) 9. Sketch the curve  $r = 2 \sin(3\theta)$  in polar coordinates.

$$0 \leq \theta \leq \frac{\pi}{6}$$

$$0 \leq 3\theta \leq \frac{\pi}{2}$$

$$0 \leq 2 \sin 3\theta \leq 2$$

$$\frac{\pi}{6} \leq \theta \leq \frac{\pi}{3}$$

$$\frac{\pi}{2} \leq 3\theta \leq \pi$$

$$1 \rightarrow 2 \sin 3\theta \rightarrow 0$$

$$\frac{\pi}{3} \leq \theta \leq \frac{\pi}{2}$$

$$\pi \leq 3\theta \leq \frac{3\pi}{2}$$

$$0 \rightarrow 2 \sin 3\theta \rightarrow -2$$

$$\frac{\pi}{2} \leq \theta \leq \frac{2\pi}{3}$$

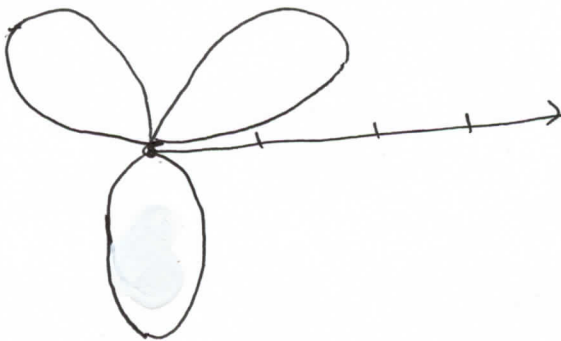
$$\frac{3\pi}{4} \leq 3\theta \leq 2\pi$$

$$-1 \leq 2 \sin 3\theta \leq 0$$

$$\frac{2\pi}{3} \leq \theta \leq \frac{5\pi}{6}$$

$$2\pi \leq 3\theta \leq \frac{5\pi}{2}$$

$$0 \leq 2 \sin 3\theta \leq 2$$



$$\frac{5\pi}{6} \leq \theta \leq \pi$$

$$\frac{5\pi}{4} \leq 3\theta \leq 3\pi$$

$$1 \rightarrow 2 \sin 3\theta \rightarrow 0$$

(10 points) 10. Maximize the function  $f(x, y, z) = x + y - z^2$  subject to the constraint  $g(x, y, z) = x^2 + y^2 + z^2 = 1$ .

$$\nabla f = \lambda \nabla g \Rightarrow \langle 1, 1, -2z \rangle = \lambda \langle 2x, 2y, 2z \rangle$$

$$(1) \quad 1 = 2\lambda x$$

$$(2) \quad 1 = 2\lambda y$$

$$(3) \quad -2z = 2\lambda z$$

$$(4) \quad x^2 + y^2 + z^2 = 1$$

$$(3) \Rightarrow 2z(-1 - \lambda) = 0 \Rightarrow z = 0 \text{ or } \lambda = -1.$$

$$\text{Case } z = 0: \text{ Then } x^2 + y^2 = 1$$

$$\text{But } x = \frac{1}{2\lambda}, y = \frac{1}{2\lambda} \text{ so } \frac{1}{4\lambda^2} + \frac{1}{4\lambda^2} = 1$$

$$1 + 1 = 4\lambda^2$$

$$\lambda = \pm \frac{1}{\sqrt{2}}$$

$$\text{Hence either } x = \frac{1}{\sqrt{2}}, y = \frac{1}{\sqrt{2}}$$

$$\text{or } x = -\frac{1}{\sqrt{2}}, y = -\frac{1}{\sqrt{2}}$$

$$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$$

$$\text{Case } \lambda = -1, \text{ Then } x = -\frac{1}{2}, y = -\frac{1}{2}$$

$$\text{So } \frac{1}{4} + \frac{1}{4} + z^2 = 1$$

$$z^2 = \frac{1}{2}, z = \pm \frac{1}{\sqrt{2}}$$

$$\text{Crit. pts. : } \left(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{\sqrt{2}}\right), \left(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{\sqrt{2}}\right),$$

$$f\left(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{\sqrt{2}}\right) = -\frac{1}{2} - \frac{1}{2} - \frac{1}{2} = -\frac{3}{2} \text{ min}$$

$$f\left(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{\sqrt{2}}\right) = -\frac{1}{2} - \frac{1}{2} - \frac{1}{2} = -\frac{3}{2} \text{ min}$$

$$f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) = \frac{2}{\sqrt{2}} = \sqrt{2} \text{ max}$$

$$f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right) = -\frac{2}{\sqrt{2}} = -\sqrt{2}$$