Math 233
Krantz

Spring, 2017
April, 2017

Third Midterm

General Instructions: Read the statement of each problem carefully. Do only what is requested—nothing more and nothing less. Of course you need not show any work for the multiple choice or the TRUE/FALSE questions. For the questions that require a written answer, provide a *complete solution.* If you only write the answer then you will not get full credit.

Be sure to ask questions if anything is unclear. This exam is worth 100 points.

(10 points) **1.** Calculate the tangent plane to the graph of the function $f(x, y) = 2xy + 3x^2$ at the point $(1, 2, 7)$.

(a) $-x + 2y + z = 7$

(b) $2x - 10y - z = 7$

(c) $10x - 2y - z = 7$

(d) $10x - 2y + z = 7$

(e) $10x + 2y - z = 7$

$\underline{n} = \langle \partial f/\partial x, \partial f/y, -1 \rangle = \langle 2y + 6x, 2x, -1 \rangle.$

At $(1,2,7)$, $\underline{n} = \langle 10, 2, -1 \rangle.$

Equation of plane is

$\langle 10, 2, -1 \rangle \cdot \langle x-1, y-2, z-7 \rangle = 0$

$(10x - 10) + (2y - 4) + (-z + 7) = 0$

$\boxed{10x + 2y - z = 7}$

(10 points) **2.** Locate and identify the local maxima and local minima and saddle points of the function $f(x,y) = y^3 + 3x^2y + 3x^2 - 15y + 6$.

(a) $(0, \sqrt{5})$ is a local min, $(0, -\sqrt{5})$ is a saddle, $(2, -1)$ is a saddle, $(-2, -1)$ is a saddle

(b) $(0, -\sqrt{5})$ is a local min, $(0, \sqrt{5})$ is a local max, $(2, -1)$ is a saddle, $(-2, -1)$ is a saddle

(c) $(0, \sqrt{5})$ is a saddle, $(0, -\sqrt{5})$ is a local max, $(2, -1)$ is a local max, $(-2, -1)$ is a saddle

(d) $(0, \sqrt{5})$ is a local max, $(0, -\sqrt{5})$ is a local max, $(2, -1)$ is a saddle, $(-2, -1)$ is a saddle

(e) $(0, \sqrt{5})$ is a local min, $(0, -\sqrt{5})$ is a saddle, $(2, -1)$ is a saddle, $(-2, -1)$ is a saddle

$\nabla f = \langle 6xy + 6x, 3y^2 + 3x^2 - 15 \rangle = \langle 0, 0 \rangle$

(1) $6xy + 6x = 0$

(2) $3y^2 + 3x^2 - 15 = 0$

(1) $\Rightarrow 6x(y+1) = 0 \Rightarrow x = 0$ or $y = -1$.

If $x = 0$ Then (2) $\Rightarrow \cancel{x}\, y = \pm\sqrt{5}$.

If $y = -1$ then (2) $\Rightarrow x = \pm 2$.

So the four critical points are $(0, \sqrt{5}), (0, -\sqrt{5}), (2, -1), (-2, -1)$.

$\mathcal{D} f = \det \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \det \begin{pmatrix} 6y+6 & 6x \\ 6x & 6y \end{pmatrix} = 6y^2 + 36y - 36x^2$

$\mathcal{D} f(0, \sqrt{5}) = 30 + 36\sqrt{5} > 0$ local min

$\mathcal{D} f(0, -\sqrt{5}) = 30 - 36\sqrt{5} < 0$ saddle

$\mathcal{D} f(2, -1) = 6 - 36 - 144 < 0$ saddle

$\mathcal{D} f(-2, -1) = 6 - 36 - 144 < 0$ saddle

(10 points) **3.** Find the extrema of the function $f(x, y) = x + y^2$ subject to the constraint $g(x, y) = x^2 + y^2 = 9$.

(a) $f(1/2, -\sqrt{35/4})$ is a minimum value of $37/4$,
$f(1/2, +\sqrt{35/4})$ is a minimum value of $37/4$,
$f(3, 0) = 3$ is not an extreme value,
$f(-3, 0) = -3$ is a minumum value of -3.

(b) $f(1/2, -\sqrt{35/4})$ is a maximum value of $37/4$,
$f(1/2, +\sqrt{35/4})$ is a maximum value of $37/4$,
$f(3, 0) = 3$ is not an extreme value,
$f(-3, 0) = -3$ is a minumum value of -3.

(c) $f(1/2, -\sqrt{35/4})$ is a maximum value of $37/4$,
$f(1/2, +\sqrt{35/4})$ is a minimum value of $37/4$,
$f(3, 0) = 3$ is a maximum,
$f(-3, 0) = -3$ is a minumum value of -3.

(d) $f(1/2, -\sqrt{35/4})$ is a minimum value of $37/4$,
$f(1/2, +\sqrt{35/4})$ is a maximum value of $37/4$,
$f(3, 0) = 3$ is not an extreme value,
$f(-3, 0) = -3$ is a minumum value of -3.

(e) $f(1/2, -\sqrt{35/4})$ is a maximum value of $37/4$,
$f(1/2, +\sqrt{35/4})$ is a maximum value of $37/4$,
$f(3, 0) = 3$ is not an extreme value,
$f(-3, 0) = -3$ is a maximum value of -3.

$\nabla f = \lambda \nabla g \Rightarrow \langle 1, 2y \rangle = \lambda \langle 2x, 2y \rangle$

$\Rightarrow 1 = 2\lambda x \quad (1) \qquad x^2 + y^2 = 9 \quad (3)$

$2y = 2\lambda y \quad (2)$

$(2) \Rightarrow 2y(1-\lambda) = 0 \Rightarrow y = 0$ or $\lambda = 1$.

Case $y = 0$: $(3) \Rightarrow x = \pm 3$ Critical pts. : $(3, 0)$, $(-3, 0)$

Case $\lambda = 1$: $(1) \Rightarrow x = \frac{1}{2}$, $(3) \Rightarrow y = \pm\sqrt{\frac{35}{4}}$ crit. pts : $(\frac{1}{2}, \sqrt{\frac{35}{4}})$, $(\frac{1}{2}, -\sqrt{\frac{35}{4}})$

$f(3, 0) = 3 \qquad f(-3, 0) = -3$

$f(\frac{1}{2}, \sqrt{\frac{35}{4}}) = \frac{1}{2} + \frac{35}{4} = \frac{37}{4} \qquad f(\frac{1}{2}, -\sqrt{\frac{35}{4}}) = \frac{1}{2} + \frac{35}{4} = \frac{37}{4}$

$\therefore (\frac{1}{2}, \sqrt{\frac{35}{4}}), (\frac{1}{2}, -\sqrt{\frac{35}{4}})$ are maxima

$(-3, 0)$ is a minimum

(10 points) **4.** Calculate the integral

$$\int_2^3 \int_1^4 x^2 y - y^2 x \, dy dx \, .$$

(a) 1/3

(b) −10/3

(c) 10/3

(d) −5

(e) 20/3

$$\int_2^3 \int_1^4 x^2 y - y^2 x \, dy \, dx = \int_2^3 \frac{x^2 y^2}{2} - \frac{y^3}{3} x \Big]_{y=1}^{y=4} dx$$

$$= \int_2^3 \left(8x^2 - \frac{64}{3}x\right) - \left(\frac{x^2}{2} - \frac{x}{3}\right) dx$$

$$= \int_2^3 \frac{15}{2}x^2 - 21x \, dx = \frac{15}{6}x^3 - \frac{21}{2}x^2 \Big]_{x=2}^{x=3}$$

$$= \left(\frac{15}{6}\cdot 27 - \frac{21}{2}\cdot 9\right) - \left(\frac{15}{6}\cdot 8 - \frac{21}{2}\cdot 4\right)$$

$$= \frac{405}{6} - \frac{189}{2} - \frac{120}{6} + \frac{84}{2}$$

$$= \frac{285}{6} - \frac{105}{2} = \frac{285}{6} - \frac{315}{6} = \frac{-30}{6} = -5.$$

(10 points) **5.** Reverse the order of integration in order to evaluate the following double integral:

$$\int_0^{\pi/2} \int_x^{\pi/2} \frac{\sin y}{y}\, dy dx\,.$$

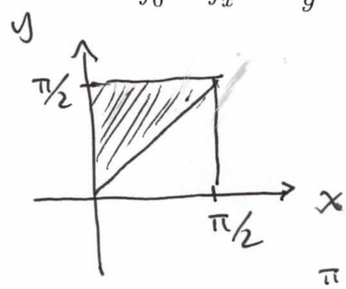
(a) -2

(b) 2

(c) 1

(d) -1

(e) 0



$$\text{integral} = \int_0^{\pi/2}\int_0^{y} \frac{\sin y}{y}\, dx\, dy = \int_0^{\pi/2} \left(\frac{\sin y}{y}\right) x \Big]_{x=0}^{x=y} dy$$

$$= \int_0^{\pi/2} \frac{\sin y}{y}\cdot y - \frac{\sin y}{y}\cdot 0\, dy = \int_0^{\pi/2} \sin y\, dy$$

$$= -\cos y\Big]_{y=0}^{y=\pi/2} = -\cos\frac{\pi}{2} + \cos 0 = 1.$$

(10 points) **6.** Calculate the volume of the solid that lies below the graph of $f(x, y) = 2x + 3y^2$ and over the rectangle $[0, 1] \times [0, 1]$ in the x-y plane.

(a) 2

(b) 1

(c) 3

(d) -2

(e) 4

$$\text{Vol.} = \int_0^1\int_0^1 2x + 3y^2\, dx\, dy$$

$$= \int_0^1 x^2 + 3xy^2\Big]_{x=0}^{x=1} dy$$

$$= \int_0^1 (1+3y^2) - (0+0)\, dy$$

$$= \int_0^1 1 + 3y^2\, dy = y + y^3\Big]_{y=0}^{y=1}$$

$$= (1+1) - (0+0) = 2.$$

(10 points) **7.** Describe the symmetries (in the x-axis, the y-axis, and the origin) of the curve $r^2 = \sin\theta$.

(a) symmeric in x-axis only

(b) symmetric in y-axis only

(c) symmetric in x-axis and y-axis

(d) symmetric in y-axis and origin

(e) symmetric in x-axis and origin

$\theta \mapsto -\theta$ $\quad r^2 = \sin(-\theta) \Rightarrow r^2 = -\sin\theta$ fails symmetry in x-axis

$\theta \mapsto \pi - \theta$ $\quad r^2 = \sin(\pi - \theta) = \sin\pi\cos(-\theta) + \cos\pi\sin(-\theta) = \sin\theta$

passes symmetry in y-axis

$r \to -r$ $\quad (-r)^2 = \sin\theta \Rightarrow r^2 = \sin\theta$ passes symmetry in origin

(10 points) **8.** What are the parametric equations of the normal line to the surface $x^2 + 4y^2 + 8z^2 = 13$ at the point $(1, 1, 1)$?

(a)

$$\begin{aligned} x &= 1 + 2t \\ y &= 1 - 8t \\ z &= 1 + 16t \end{aligned}$$

(b)

$$\begin{aligned} x &= 1 - 2t \\ y &= 1 + 8t \\ z &= 1 + 16t \end{aligned}$$

(c)

$$\begin{aligned} x &= 1 + 2t \\ y &= 1 + 8t \\ z &= 1 + 16t \end{aligned}$$

$\underline{n} = \langle 2x, 8y, 16z \rangle$.

At the point $(1, 1, 1)$, $\underline{n} = \langle 2, 8, 16 \rangle$.

Parametric equations are

$x = 1 + 2t$

$y = 1 + 8t$

$z = 1 + 16t$

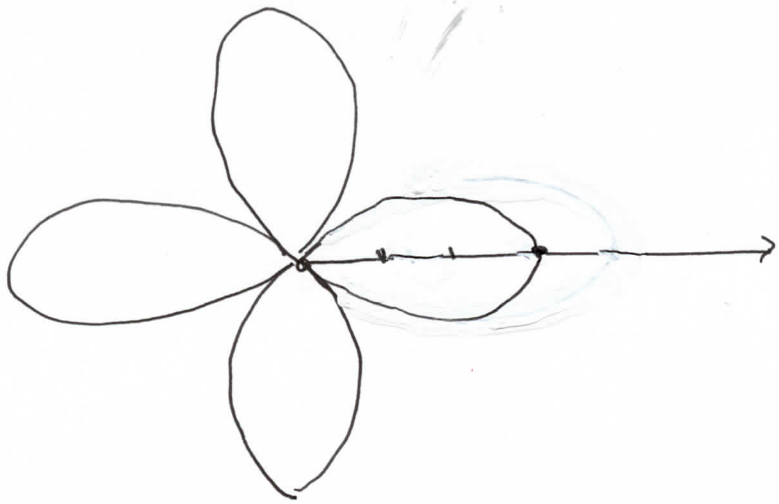
(d)

$$
\begin{aligned}
x &= 1 + 2t \\
y &= 1 + 8t \\
z &= 1 - 16t
\end{aligned}
$$

(e)

$$
\begin{aligned}
x &= 1 + 2t \\
y &= 1 + 4t \\
z &= 1 + 16t
\end{aligned}
$$

(10 points) **9.** Sketch the curve $r = 3\cos(2\theta)$ in polar coordinates.



10 points) **10.** Maximize the function $f(x, y, z) = x + 2y + z^2$ subject to the constraint $g(x, y, z) = x^2 + y^2 + z^2 = 4$.

$$\nabla f = \lambda \nabla g$$

$$\langle 1, 2, 2z \rangle = \lambda \langle 2x, 2y, 2z \rangle$$

(1) $1 = 2\lambda x$

(2) $2 = 2\lambda y$

(3) $2z = 2\lambda z$

(4) $x^2 + y^2 + z^2 = 4$

(3) $\Rightarrow 2z(1-\lambda) = 0 \Rightarrow z = 0$ or $\lambda = 1$

$\underline{z = 0}$: (4) $\Rightarrow x^2 + y^2 = 4$ (1) $\Rightarrow x = \frac{1}{2\lambda}$

(2) $\Rightarrow y = \frac{1}{\lambda}$

So $\frac{1}{4\lambda^2} + \frac{1}{\lambda^2} = 4$

$\frac{5}{4} \cdot \frac{1}{\lambda^2} = 4 \Rightarrow \lambda^2 = \frac{5}{16} \Rightarrow \lambda = \pm \frac{\sqrt{5}}{4}$.

Thus either $x = \frac{2}{\sqrt{5}}$, $y = \frac{4}{\sqrt{5}}$

or $x = -\frac{2}{\sqrt{5}}$, $y = \frac{-4}{\sqrt{5}}$

In either case, (4) $\Rightarrow z = 0$. Crit pts: $\left(-\frac{2}{\sqrt{5}}, \frac{-4}{\sqrt{5}}, 0\right), \left(\frac{2}{\sqrt{5}}, \frac{4}{\sqrt{5}}, 0\right)$

$\underline{\lambda = 1}$: Then $x = \frac{1}{2}$, $y = 1$. (3) $\Rightarrow z = \pm \frac{\sqrt{11}}{2}$.

Crit pts: $\left(\frac{1}{2}, 1, \frac{\sqrt{11}}{2}\right), \left(\frac{1}{2}, 1, -\frac{\sqrt{11}}{2}\right)$.

$f\left(-\frac{2}{\sqrt{5}}, \frac{-4}{\sqrt{5}}, 0\right) = -\frac{10}{\sqrt{5}} = -2\sqrt{5}$ $f\left(\frac{2}{\sqrt{5}}, \frac{4}{\sqrt{5}}, 0\right) = \frac{10}{\sqrt{5}} = 2\sqrt{5}$

$f\left(\frac{1}{2}, 1, \frac{\sqrt{11}}{2}\right) = \frac{5}{2} + \frac{11}{4} = \frac{21}{4}$ $f\left(\frac{1}{2}, 1, -\frac{\sqrt{11}}{2}\right) = \frac{5}{2} + \frac{11}{4} = \frac{21}{4}$.

So $\left(\frac{1}{2}, 1, \frac{\sqrt{11}}{2}\right), \left(\frac{1}{2}, 1, -\frac{\sqrt{11}}{2}\right)$ are maxima

$\left(-\frac{2}{\sqrt{5}}, \frac{-4}{\sqrt{5}}, 0\right)$ is a minimum.