Math 310
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Figure: This is your instructor.

Notice that we can say that A $\Leftrightarrow$ B is true only when both $A \Rightarrow B$ and $B \Rightarrow A$ are true. An examination of the truth table reveals that $A \Leftrightarrow B$ is true precisely when A and B are either both true or both false. Thus $A \Leftrightarrow B$ means precisely that A and B are logically equivalent. One is true when and *only when* the other is true. One is false when and *only when* the other is false.

Example

The statement

$$x > 0 \Leftrightarrow 2x > 0$$

is true. For if $x > 0$, then $2x > 0$; and if $2x > 0$, then $x > 0$.

Example

The statement

$$x > 0 \Leftrightarrow x^2 > 0$$

is false. For $x > 0 \Rightarrow x^2 > 0$ is certainly true, while $x^2 > 0 \Rightarrow x > 0$ is false ($(-3)^2 > 0$ but $-3 \not> 0$).

Example

The statement

$$\{\sim (A \vee B)\} \Leftrightarrow \{(\sim A) \wedge (\sim B)\} \qquad (*)$$

is true because the truth table for $\sim(\mathrm{A} \vee \mathrm{B})$ and that for $(\sim \mathrm{A}) \wedge (\sim \mathrm{B})$ are the same. Thus they are logically equivalent: one statement is true precisely when the other is. Another way to see the truth of $(*)$ is to examine the truth table for the full statement:

A	B	$\sim (A \vee B)$	$(\sim A) \wedge (\sim B)$	$\{\sim (A \vee B)\} \Leftrightarrow \{(\sim A) \wedge (\sim B)\}$
T	T	F	F	T
T	F	F	F	T
F	T	F	F	T
F	F	T	T	T

Given an implication

$$A \Rightarrow B,$$

the *contrapositive* statement is defined to be the implication

$$\sim B \Rightarrow \sim A.$$

The contrapositive is logically equivalent to the original implication, as we see by examining their truth tables:

A	B	$A \Rightarrow B$
T	T	T
T	F	F
F	T	T
F	F	T

and

A	B	$\sim A$	$\sim B$	$(\sim B) \Rightarrow (\sim A)$
T	T	F	F	T
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

Example

The statement

If it is raining, then it is cloudy.

has, as its contrapositive, the statement

If there are no clouds, then it is not raining.

A moment's thought convinces us that these two statements say the same thing: if there are no clouds, then it could not be raining; for the presence of rain implies the presence of clouds.

The main point to keep in mind is that, given an implication $A \Rightarrow B$, its *converse* $B \Rightarrow A$ and its *contrapositive* $(\sim B) \Rightarrow (\sim A)$ are entirely different statements. The converse is distinct from, and *logically independent from*, the original statement. The contrapositive is distinct from, but *logically equivalent to*, the original statement.

Some classical treatments augment the concept of *modus ponendo ponens* with the idea of *modus tollendo tollens*. It is in fact logically equivalent to *modus ponendo ponens*. *Modus tollendo tollens* says

If $\sim B$ *and* $A \Rightarrow B$ *then* $\sim A$.

It is common to abbreviate *modus ponendo ponens* by *modus ponens* and *modus tollendo tollens* by *modus tollens*.

Modus tollens actualizes the fact that $(\sim B) \Rightarrow (\sim A)$ is logically equivalent to $A \Rightarrow B$. The first of these implications is of course the *contrapositive* of the second.

Quantifiers

The mathematical statements that we will encounter in practice will use the *connectives* "and," "or," "not," "if-then," and "iff." They will also use *quantifiers*. These two basic quantifiers are "for all" and "there exists."

Example

Consider the statement

All automobiles have wheels.

This statement makes an assertion about *all* automobiles. It is true because every automobile does have wheels.

Compare this statement with the next one:

There exists a woman who is blonde.

This statement is of a different nature. It does not claim that all women have blonde hair—merely that there exists *at least one* woman who does. Since that is true, the statement is true.

Example

Consider the statement

All positive real numbers are integers.

This sentence asserts that something is true for all positive real numbers. It is indeed true for *some* positive numbers, such as 1 and 2 and 193. However, it is false for at least one positive number (such as $1/10$ or π), so the entire statement is false.

Here is a more extreme example:

The square of any real number is positive.

This assertion is *almost* true—the only exception is the real number 0: $0^2 = 0$ is not positive. But it only takes one exception to falsify a "for all" statement. So the assertion is false.

This last example illustrates the principle that the negation of a "for all" statement is a "there exists" statement.

Example

Look at the statement

There exists a real number which is greater than 5.

In fact, there are lots of numbers that are greater than 5; some examples are $7, 42, 2\pi$, and $97/3$. Other numbers, such as 1, 2, and $\pi/6$, are not greater than 5. Since there is *at least one* number satisfying the statement, the assertion is true.

Example

Consider the statement

There is a man who is at least 10 feet tall.

This statement is false. To *verify* that it is false, we must demonstrate that *there does not exist a man who is at least 10 feet tall*. In other words, we must show that all men are shorter than 10 feet.

The negation of a "there exists" statement is a "for all" statement.

A somewhat different example is the sentence

There exists a real number x which satisfies the equation

$$x^3 - 2x^2 + 3x - 6 = 0 .$$

There is in fact only one real number that satisfies the equation, and that is $x = 2$. Yet that information is sufficient to show that the statement true.

We often use the symbol $\forall$ to denote "for all" and the symbol $\exists$ to denote "there exists." The assertion

$$\forall x \, , \; x + 1 < x$$

claims that for every x, the number $x + 1$ is less than x. If we take our universe to be the standard real number system, then this statement is false. The assertion

$$\exists x \, , \; x^2 = x$$

claims that there is a number whose square equals itself. If we take our universe to be the real numbers, then the assertion is satisfied by $x = 0$ and by $x = 1$. Therefore, the assertion is true.

In all the examples of quantifiers that we have discussed thus far, we were careful to specify our *universe*. That is, "There is a woman such that ..." or "All positive real numbers are ..." or "All automobiles have ...". The quantified statement makes no sense unless we specify the universe of objects from which we are making our specification. In the discussion that follows, we will always interpret quantified statements in terms of a universe. Sometimes the universe will be explicitly specified, while other times it will be understood from context.

Quite often we will encounter $\forall$ and $\exists$ used together. The following examples are typical:

Example

The statement

$$\forall x\ \exists y\,,\ \ y > x$$

claims that for any number x there is a number y that is greater than it. In the realm of the real numbers, this is true. In fact, $y = x + 1$ will always do the trick.

The statement

$$\exists x\ \forall y\,,\ \ y > x$$

has quite a different meaning from the first one. It claims that there is an x that is less than *every* y. This is absurd. For instance, x is *not* less than $y = x - 1$.

Example

The statement

$$\forall x \; \forall y \, , \; x^2 + y^2 \geq 0$$

is true in the realm of the real numbers: it claims that the sum of two squares is always greater than or equal to zero. [This statement happens to be *false* in the realm of the complex numbers. We shall learn about that number system later. When we interpret a logical statement, it will always be important to understand the context, or universe, in which we are working.]

The statement

$$\exists x \; \exists y \, , \; x + 2y = 7$$

is true in the realm of the real numbers: it claims that there exist x and y such that $x + 2y = 7$. Certainly the numbers $x = 3, y = 2$ will do the job (although there are many other choices that work as well).

We conclude by noting that $\forall$ and $\exists$ are closely related. The statements

$$\forall x\,,\ A(x) \qquad \text{and} \qquad \sim \exists x\,,\ \sim A(x)$$

are logically equivalent. The first asserts that the statement $A(x)$ is true for all values of x. The second asserts that there exists no value of x for which $A(x)$ fails, which is the same thing.

Likewise, the statements

$$\exists x\,, B(x) \qquad \text{and} \qquad \sim \forall x\,, \sim B(x)$$

are logically equivalent. The first asserts that there is some x for which $B(x)$ is true. The second claims that it is not the case that $B(x)$ fails for every x, which is the same thing. The books [HALM] and [GIH] explore the algebraic structures inspired by these quantifiers.

The assertions

$$\forall x \, , A(x) \Leftrightarrow \sim \exists x \, , \sim A(x)$$

and

$$\exists x \, , B(x) \Leftrightarrow \sim \forall x \, , \sim B(x)$$

are commonly referred to as *de Morgan's Laws*. You should compare them with the de Morgan Laws that we discussed in an earlier lecture.

It is worth noting explicitly that $\forall$ and $\exists$ do *not* commute. That is to say,

$$\forall x \exists y \, , F(x, y) \qquad \text{and} \qquad \exists y \forall x \, , \; F(x, y)$$

do *not* say the same thing. We invite you to provide a counterexample.

A "for all" statement is something like the conjunction of a very large number of simpler statements. For example, the statement

$$\text{For every nonzero integer } n, \; n^2 > 0 \, .$$

is actually an efficient way of saying that $1^2 > 0$ and $(-1)^2 > 0$ and $2^2 > 0$, etc. It is not feasible to apply truth tables to "for all" statements, and we usually do not do so.

A "there exists" statement is something like the disjunction of a very large number of statements (the word "disjunction" in the present context means an "or" statement). For example, the statement

$$\text{There exists an integer } n \text{ such that } P(n) = 2n^2 - 5n + 2 = 0 \,.$$

is actually an efficient way of saying that $P(1) = 0$ or $P(-1) = 0$ or $P(2) = 0$, etc. It is not feasible to apply truth tables to "there exist" statements, and we usually do not do so.

It is common to say that *first-order logic* consists of the connectives $\wedge$, $\vee$, $\sim$, $\Rightarrow$, $\iff$, the equality symbol $=$, and the quantifiers $\forall$ and $\exists$, together with an infinite string of variables $x, y, z, \ldots, x', y', z', \ldots$ and, finally, parentheses (, ,) to keep things readable (see [BAR, p. 7]). The word "first" here is used to distinguish the discussion from second-order and higher-order logics. In first-order logic, the quantifiers $\forall$ and $\exists$ always range over elements of the domain M of discourse. Second-order logic, by contrast, allows us to quantify over subsets of M and functions F mapping $M \times M$ into M. Third-order logic treats sets of function and more abstract constructs. The distinction among these different orders is often moot.